XII. NON-RENORMALIZATION THEOREMS

As we briefly discussed in the introduction to the course, an important feature of supersymmetry is the fact that it leads to cancellation of divergences in loop diagrams. For instance, this is the reason that suggested to use susy generalizations of the Standard Model of electroweak interactions to solve the problem of the instability of the Higgs mass against radiative corrections. In this section we will discuss more extensively the properties of susy theories from the point of view of renormalization. We will see that the behaviour of these theories under renormalization is extremely simple and the "finiteness" of the theories improves for extended susy theories leading to the result that $N=4$ theories are free of ultraviolet divergences.

• Non-renormalization theorems in $N=1$.

In this section we discuss a method proposed by Seiberg, which showed how the non-renormalization theorems may be obtained by simple symmetry considerations.

We consider a general renormalizable susy theory with chiral superfields Φ_n and gauge superfields V_A. The Lagrangian is of the form

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\ \bar\Phi\, e^{2V}\Phi + \int d^2\theta\ W(\Phi) + \frac{1}{4g^2}\int d^2\theta\ W_A^\alpha W_{A\alpha} + h.c. , \qquad (*)$$

where $W(\Phi)$ is a cubic polynomial and we ignored possible θ_{YM}-terms, which have no effect in perturbation theory.

We will see that, under radiative corrections

- The Kahler potential is renormalized at all loop orders,
- The kinetic term for the gauge fields is renormalized only at one-loop,
- The superpotential $W(\Phi)$ is not renormalized.

To prove the theorem we reinterpret this theory as a special case of a theory with two additional external gauge-invariant chiral superfields

$$X = (x, \psi_x, F_x) \quad , \quad Y = (y, \psi_y, F_y)$$

with Lagrangian

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\ \bar\Phi\, e^{2V}\Phi + \int d^2\theta\ Y W(\Phi) + \frac{1}{4}\int d^2\theta\ X W_A^\alpha W_{A\alpha} + h.c.$$

This Lagrangian becomes equal to the original one when the scalar components of the extra fields are taken to be

$$x = \frac{1}{g^2} \quad , \quad y = 1 ,$$

and the spinors $\psi_{x,y}$ and the auxiliary fields $F_{x,y}$ are set equal to zero.

To analyse the renormalization properties of the theory we impose a cut-off λ on the momenta in the loops. We can find a local Wilsonian effective Lagrangian $\mathcal{L}_\lambda$ by integrating out all the momenta above λ

$$\exp\left(i\int d^4x\, \mathcal{L}_\lambda\right) = \int_{|p|>\lambda} \mathcal{D}\varphi\, \exp\left(i\int d^4x\, \mathcal{L}\right).$$

The effective Lagrangian gives exactly the same results as the original one for S-matrix elements of processes at momenta below λ. We want to study the form of this effective Lagrangian. If we consider a cut-off procedure which preserves supersymmetry and gauge-invariance we get

$$\mathcal{L}_\lambda = \int d^2\theta\, d^2\bar\theta\;\; J(\Phi, \bar\Phi, V, X, Y, D_\alpha \cdots) + \int d^2\theta\;\; H(\Phi, X, Y, W^\alpha) + h.c. \qquad (**)$$

with J and H both gauge-invariant.

The dependence on X and Y is severely limited by two additional symmetries of the action (these symmetries are only broken by non perturbative effects).
The first symmetry is a $U(1)$ R-symmetry under which Φ, V and X are uncharged, while Y has charge 2 (notice that, given that θ has R-charge -1 and $\bar\theta$ has R-charge $+1$, W^α has R-charge $+1$ because it is obtained by acting with $\bar D^2 D_\alpha$ on V), summarizing

	Φ_m	V	X	Y	θ	$\bar\theta$	W^α
R-charge	0	0	0	+2	−1	+1	+1

Given that $H(\Phi, X, Y, W^\alpha)$ must be chiral, it must be an <u>holomorphic</u> function of its arguments, so it can be only of first order in Y or of second order in W^α, with coefficients depending on the R-neutral superfields Φ and/or X:

$$\underline{H(\Phi, X, Y, W^\alpha) = Y\, W_\lambda(X, \Phi) + h_{\lambda AB}(X, \Phi)\, W^\alpha_A W_{B\alpha}}.$$

The second symmetry is the translation of X by an imaginary constant

$$X \to X + i\xi$$

with ξ real. This changes the Lagrangian by an amount proportional to $\mathrm{Im}(W^\alpha_A W_{A\alpha})$, which is a total derivative and, hence, has no effect in perturbation theory. This symmetry prevents X from appearing anywhere in the effective Lagrangian $(**)$ except where it appears in the original Lagrangian $(*)$. So we conclude that W_λ is independent of X while $h_{\lambda AB}$ has the form

$$h_{\lambda AB} = c_\lambda\, \delta_{AB}\, X + l_{\lambda AB}(\Phi),$$

where c_λ is a real (cut-off dependent) constant.

Writing the terms explicitly

$$\underline{H(\Phi, X, Y, W^\alpha) = Y W_2(\Phi) + \left(c_2 \delta_{AB} X + \ell_{2AB}(\Phi)\right) W^\alpha_A W_{B\alpha}} \, .$$

We can now determine the coefficients of the effective Lagrangian by setting the auxiliary superfields X and Y to suitable values and then using perturbation theory where it is trustable. In particular we set the spinor and auxiliary components of X and Y to zero and we take the limits

$$x \to \infty$$
$$y \to 0$$

so that the gauge coupling constant vanishes as $1/\sqrt{x}$ and all Yukawa and scalar couplings derived from the superpotential vanish as y.

In this limit the only diagram which contributes to the term $Y W_2(\Phi)$ has a single vertex arising from the term $\int d^2\theta \; Y W(\Phi) + h.c.$ in eq. $(*)$, so we get

$$\underline{W_2(\Phi) = W(\Phi)} \, .$$

Moreover, with $Y=0$, there is a conservation law which requires every term in L_2 to have an equal number of Φ and $\bar{\Phi}$, so, since $\bar{\Phi}$ can not appear in ℓ_{2AB} neither can Φ. Gauge invariance then requires for a simple gauge group:

$$\underline{\ell_{2AB} = \delta_{AB} L_2} \, .$$

Let's now count the degree of divergence of the diagrams. We have the following scaling in x:

$$\begin{cases} \text{gauge propagators} & \sim \frac{1}{x} \\ \text{pure gauge interactions} & \sim x \\ \text{scalar propagators} & \sim 1 \\ \text{scalar interactions} & \sim 1 \end{cases}$$

With $y=0$ the number of powers of x in a diagram with V_W pure gauge boson vertices, I_W internal gauge boson lines and any number of scalar-gauge boson vertices and scalar propagators is

$$N_x = V_W - I_W \, .$$

The number of loops is given by

$$L = I_W + I_\Phi - V_W - V_\Phi + 1 \, ,$$

where I_Φ is the number of internal Φ lines and V_Φ is the number of Φ-V interaction vertices. All the Φ-V vertices have two Φ lines attached, so with no external Φ lines $I_\Phi = V_\Phi$, so that

$$L = I_W - V_W + 1 \, ,$$

and we get

$$\underline{N_X = 1 - L\,.}$$

Terms linear in X come only from the tree-level diagrams $(L=0)$, so the coefficient c_2 in the effective Lagrangian is correctly given by the tree approximation, and therefore it is equal to its value in the original Lagrangian

$$\underline{c_2 = 1\,.}$$

On the other hand, the coefficient L_2 of the X-independent term is given by one-loop diagrams $(L=1)$ only.

Putting everything together

$$\mathcal{L}_2 = \int d^2\theta\, d^2\bar\theta \;\; \Sigma(\Phi, \bar\Phi, V, X, Y, D_\alpha, \dots)$$
$$+ \int d^2\theta\; Y\, W(\Phi) + \frac{1}{4}\int d^2\theta\, (X + L_2)\, W^\alpha_A W_{A\alpha} + h.c.$$

where L_2 is the one-loop contribution. Setting $Y=1$ and $X = 1/g^2$ we obtain

$$\underline{\mathcal{L}_2 = \int d^2\theta\, d^2\bar\theta \;\; \Sigma(\Phi, \bar\Phi, V, X, Y, D_\alpha, \dots)}$$
$$\underline{+ \int d^2\theta\; W(\Phi) + \frac{1}{4 g_2^2}\int d^2\theta\; W^\alpha_A W_{A\alpha} + h.c.}$$

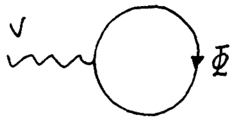
where the renormalized gauge coupling is $\underline{g_2^{-2} = g^{-2} + L_2}$ and is corrected <u>only</u> at one-loop. The Kahler potential is instead renormalized at all orders.

· <u>The Fayet-Iliopoulos term</u>

We have seen that in the presence of $U(1)$ gauge factors we can also introduce a Fayet-Iliopoulos term in the Lagrangian

$$\mathcal{L}_{FI} = \int d^2\theta\, d^2\bar\theta\;\; \xi V\,.$$

It can be shown that the radiative corrections to this term come only from tadpole diagrams in which a single external gauge line is attached to a chiral loop.



The contribution from these diagrams is proportional to the sum of the gauge couplings of all chiral superfields, that is to the trace of the $U(1)$ generator. But this trace must vanish in order to avoid gravitational anomalies, so we get that

· the Fayet-Iliopoulos term is <u>not renormalized</u>.

· Comments

The above results on the non-renormalization properties of susy theories can be extended to non-renormalizable theories as well. In such theories the $\int d^2\theta d^2\bar\theta\, \bar\Phi e^{2V}\Phi$ term is replaced by a general Kahler potential $\int d^2\theta d^2\bar\theta\, K(\bar\Phi e^{2V}, \Phi)$, while the superpotential and the gauge kinetic term are replaced by an arbitrary globally gauge-invariant scalar function $f(\Phi, W_\alpha)$. It can be shown that

· $f_\Lambda(\Phi, W_\alpha)$ is the same as $f(\Phi, W_\alpha)$ to all orders in perturbation theory, except for the one loop renormalization of the term quadratic in W_α.

· An important application of the non-renormalization theorem is related to supersymmetry breaking. It can be shown that, if there are no Fayet-Iliopoulos terms and if the superpotential $W(\Phi)$ allows solutions of the equations $\frac{\partial W(\Phi)}{\partial \Phi_n} = 0$, then supersymmetry is not broken in any finite order of perturbation theory.

(see Weinberg III section 27.6. for more details.)

· Non-renormalization theorems in extended supersymmetry

For theories with extended rigid supersymmetry even stronger non-renormalization properties are found.

· In N=2 theories the full perturbative action does not contain any corrections for more than one loop.

This result has also the consequence that, if an N=2 theory is finite at one-loop level, then it is finite to all orders.

· N=4 theories are finite to all orders in perturbation theory. These theories also have a conformal invariance and their β function $(\beta(g) = \mu\, dg/d\mu)$ is zero:

$$\underline{\beta(g) = 0}.$$