X. THE SUPERCURRENT.

Like any continuous global symmetry, supersymmetry leads to the existence of a conserved current. The conservation and commutation properties of the supersymmetry current are operator equations that will remain valid even when supersymmetry is spontaneously broken.

In the presence of an ordinary global symmetry of the Lagrangian density under an infinitesimal transformation

$$\chi^\ell \to \chi^\ell + \delta\chi^\ell \equiv \chi^\ell + \varepsilon \mathcal{F}^\ell ,$$

where ε is an infinitesimal parameter (χ^ℓ is a generic canonical or auxiliary field and $\mathcal{F}^\ell$ is a function of the canonical and auxiliary fields) leads to the existence of a Noether current

$$j^\mu = \sum_\ell \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi^\ell)} \cdot \mathcal{F}^\ell .$$

This current is conserved for fields satisfying the field equations

$$\partial_\mu \left(\frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^\ell)} \right) = \frac{\partial_L \mathcal{L}}{\partial \chi^\ell} ,$$

notice that here we chose the convention to take left-derivatives when we are deriving with respect to a fermion field. The spatial integral of the conserved supercurrent density, j^0, gives us the generators of the symmetry

$$Q = \int d^3x \; j^0(x) ,$$

whose commutators computed by using the canonical commutation relations give

$$\left[\int d^3x \, j^0(x) , \; \chi^\ell \right] = \mathcal{F}^\ell(x) .$$

The supersymmetry current requires a somewhat more complicated treatment because supersymmetry is <u>only</u> a symmetry of the action and not of the Lagrangian. The variation of the Lagrangian under a susy transformation is a spacetime derivative

$$\underline{\delta \mathcal{L} = \varepsilon^\alpha \partial_\mu K^\mu_\alpha + \partial_\mu \bar{K}^\mu_{\dot\alpha} \bar{\varepsilon}^{\dot\alpha}}$$

with K^μ_α (and $\bar{K}^\mu_{\dot\alpha}$) four-vectors whose components are spinors. If we compute the usual Noether current we get

$$\underline{\sum_\ell \delta\chi^\ell \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^\ell)} \equiv - \varepsilon^\alpha N^\mu_\alpha - \bar{N}^\mu_{\dot\alpha} \bar{\varepsilon}^{\dot\alpha}} \qquad (*)$$

whose divergence is given by

$$\begin{aligned}
\varepsilon^\alpha \partial_\mu N^\mu_\alpha + \partial_\mu \bar{N}^\mu_{\dot\alpha} \bar{\varepsilon}^{\dot\alpha} &= - \sum_\ell (\partial_\mu \delta\chi^\ell) \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^\ell)} - \sum_\ell \delta\chi^\ell \partial_\mu \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^\ell)} \\
&= - \sum_\ell (\partial_\mu \delta\chi^\ell) \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^\ell)} - \sum_\ell \delta\chi^\ell \frac{\partial_L \mathcal{L}}{\partial \chi^\ell} \\
&= - \delta \mathcal{L} .
\end{aligned}$$

To obtain a conserved quantity, we need to define the supersymmetry current as

$$\underline{S^\mu \equiv N^\mu + K^\mu},$$

which, according to the above results satisfies

$$\underline{\partial_\mu S^\mu = 0}\,.$$

The generators of susy transformations can be expressed as

$$Q_\alpha = \int d^3x\, S^0_\alpha\,, \qquad \bar{Q}_{\dot\alpha} = \int d^3x\, \bar{S}^0_{\dot\alpha}\,.$$

The variation of a field X under susy transformations is given, as usual, by

$$[i\varepsilon^\alpha Q_\alpha + i\bar{Q}_{\dot\alpha}\bar{\varepsilon}^{\dot\alpha}, X] = \delta X\,.$$

· As an example we can compute the supercurrent for the massless Wess-Zumino model. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu z)^\dagger \partial^\mu z - i\psi\sigma^\mu\partial_\mu\bar{\psi} + F^\dagger F\,.$$

The susy transformations for the components of the chiral superfield are

$$\begin{cases} \delta z = \sqrt{2}\,\varepsilon\psi \\ \delta\psi_\alpha = -\sqrt{2}\,\varepsilon_\alpha F + \sqrt{2}\, i\,\sigma^\mu_{\alpha\dot\beta}\,\bar{\varepsilon}^{\dot\beta}\,\partial_\mu z \\ \delta F = \sqrt{2}\, i\,\partial_\mu\psi\,\sigma^\mu\bar{\varepsilon} \end{cases}$$

By a straightforward computation we get

$$\begin{aligned} \varepsilon^\alpha N^\mu_\alpha + \bar{N}^\mu_{\dot\alpha}\bar{\varepsilon}^{\dot\alpha} &= -\delta z^\dagger\partial^\mu z - \delta z\,\partial^\mu z^\dagger - i\,\delta\bar{\psi}\,\bar{\sigma}^\mu\psi \\ &= -\sqrt{2}\,\varepsilon\psi\,\partial^\mu z^\dagger - \sqrt{2}\,\bar{\varepsilon}\bar{\psi}\,\partial^\mu z - i\sqrt{2}\,\psi\sigma^\mu\bar{\varepsilon}F^\dagger - \sqrt{2}\,\varepsilon\sigma^\nu\bar{\sigma}^\mu\psi\,\partial_\nu z^\dagger \end{aligned}$$

and

$$\begin{aligned} \delta\mathcal{L} &= (\partial_\mu\delta z)^\dagger\partial^\mu z + (\partial_\mu z)^\dagger\partial^\mu\delta z + i\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\delta\psi + i(\partial_\mu\delta\bar{\psi})\,\bar{\sigma}^\mu\psi + \delta F^\dagger F + F^\dagger\delta F \\ &= \sqrt{2}\,\partial_\mu\left[\varepsilon\psi\,\partial^\mu z^\dagger + iF^\dagger\psi\sigma^\mu\bar{\varepsilon} - 2\bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\varepsilon}\,\partial_\nu z\right] \end{aligned}$$

which implies

$$\varepsilon^\alpha K_\alpha + \bar{K}_{\dot\alpha}\bar{\varepsilon}^{\dot\alpha} = \sqrt{2}\,\varepsilon\psi\,\partial^\mu z^\dagger + \sqrt{2}\, i\,\psi\sigma^\mu\bar{\varepsilon}F^\dagger - 2\sqrt{2}\,\bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\varepsilon}\,\partial_\nu z\,.$$

The supercurrent is thus given by

$$\begin{aligned} \varepsilon S^\mu + \bar{S}^\mu\bar{\varepsilon} &= \varepsilon N^\mu + \bar{N}^\mu\bar{\varepsilon} + \varepsilon K^\mu + \bar{K}^\mu\bar{\varepsilon} \\ &= -\sqrt{2}\,\varepsilon\sigma^\nu\bar{\sigma}^\mu\psi\,\partial_\nu z^\dagger - \sqrt{2}\,\bar{\psi}\bar{\sigma}^\mu\sigma^\nu\bar{\varepsilon}\,\partial_\nu z \end{aligned}$$

or, explicitly:

$$\begin{cases} S^\mu = -\sqrt{2}\,\sigma^\nu\bar{\sigma}^\mu\psi\,\partial_\nu z^\dagger \\ \bar{S}^\mu = -\sqrt{2}\,\bar{\psi}\bar{\sigma}^\mu\sigma^\nu\,\partial_\nu z \end{cases}$$

There is another definition of symmetry currents in terms of the response of the action to a local symmetry transformation. In the absence of supergravity the action is not invariant under local susy transformations. If we make such a transformation with a spacetime-dependent parameter $\varepsilon(x)$, the action will change by an amount that, in order to vanish when $\varepsilon(x)$ is constant, must (even when the field equations are not satisfied) be of the form

$$\delta I = -\int d^4x \left[(\partial_\mu \varepsilon(x)) S^\mu(x) + \bar{S}^\mu(x)(\partial_\mu \bar{\varepsilon}(x)) \right], \qquad (**)$$

where $S^\mu(x)$ is a four-vector of fermion operators. To completely define S^μ we need to generalize the global susy transformations to a local version. There is a choice of this generalization which guarantees that S^μ coincide with the previously computed supersymmetry current. This choice is to impose that derivatives of $\varepsilon(x)$ do not appear in the transformation of the canonical or auxiliary fields. For example, for chiral superfields

$$\begin{cases} \delta z = \sqrt{2}\, \varepsilon(x) \psi \\ \delta \psi_\alpha = -\sqrt{2}\, \varepsilon_\alpha(x) F + \sqrt{2}\, i\, \sigma^\mu_{\alpha\dot\beta}\, \bar{\varepsilon}^{\dot\beta}(x)\, \partial_\mu z \\ \delta F = \sqrt{2}\, i (\partial_\mu \psi) \sigma^\mu \bar{\varepsilon}(x) \end{cases}$$

With this choice the change of the action under the local susy transformation has two pieces. The first piece comes from the terms containing derivatives of the canonical fields. When we take their variation we will get a contribution proportional to $\partial_\mu \varepsilon(x)$. One can easily check that (compare eq. (*))

$$\underline{\delta_1 S = -\int d^4x \left[(\partial_\mu \varepsilon(x)) N^\mu + \bar{N}^\mu (\partial_\mu \bar{\varepsilon}(x)) \right]}.$$

The second term arises from the fact that the Lagrangian density is not invariant even under the part of the susy transformation that does not involve derivatives of $\varepsilon(x)$. In this case we simply get

$$\underline{\delta_2 S = \int d^4x \left[\varepsilon(x)\, \partial_\mu K^\mu + (\partial_\mu \bar{K}^\mu) \bar{\varepsilon}(x) \right] = -\int d^4x \left[(\partial_\mu \varepsilon(x)) K^\mu + \bar{K}^\mu (\partial_\mu \bar{\varepsilon}(x)) \right]}.$$

Adding the two contributions $\delta_1 S$ and $\delta_2 S$ we get a total change

$$\delta S = -\int d^4x \left[(\partial_\mu \varepsilon(x))(N^\mu + K^\mu) + (\bar{N}^\mu + \bar{K}^\mu)(\partial_\mu \bar{\varepsilon}(x)) \right]$$

which shows that S^μ defined by (**) coincides with the supersymmetry current.

Some remarks on the supercurrent.

The above definitions of the supercurrent do not uniquely specify the form of the supercurrent, because we can always introduce a modified current

$$S^\mu_{NEW} \equiv S^\mu + \partial_\nu A^{\mu\nu}$$

with $A^{\mu\nu} = -A^{\nu\mu}$ an arbitrary antisymmetric tensor of spinors. The term $\partial_\nu A^{\mu\nu}$ is conserved whether or not the field equations are satisfied ($\partial_\mu \partial_\nu A^{\mu\nu} = 0$ because of the antisymmetry property), and its time component is a space derivative, so

$$\int d^3x \, S^0_{NEW} = \int d^3x \, S^0$$

gives the same supercharge.

There is a particular choice of $A^{\mu\nu}$ with the feature that $\bar\sigma_\mu S^\mu_{NEW}$ turns out to be a measure of the violation of scale invariance in the theory. This is attained in the Wess-Zumino model by choosing

$$A^{\mu\nu} = -\frac{\sqrt{2}}{3}\left(\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu\right) \psi z^+$$

Let's compute explicitly the case with only a kinetic term, which is obviously scale-invariant. We get

$$\bar\sigma_\mu S^\mu = 2\sqrt{2}\, \bar\sigma^\nu \psi \, \partial_\nu z^+$$

and using the equation of motion for ψ (that is $\bar\sigma^\nu \partial_\nu \psi = 0$), one gets

$$\bar\sigma_\mu S^\mu = 2\sqrt{2}\, \bar\sigma^\nu \partial_\nu (\psi z^+).$$

We also have

$$\bar\sigma_\mu \partial_\nu A^{\mu\nu} = -\frac{\sqrt{2}}{3} \bar\sigma_\mu \left(\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu\right) \partial_\nu (\psi z^+)$$
$$= -2\sqrt{2}\, \bar\sigma^\nu \partial_\nu (\psi z^+)$$

that is

$$\underline{\bar\sigma_\mu S^\mu_{NEW} = 0},$$

in agreement with the scale-invariance of the theory.

- Another remarkable property of the supercurrent is the fact that it can also be interpreted as one of the components of a "supercurrent supermultiplet". This supermultiplet, in addition to the supercurrent, also contains the energy-momentum tensor $T^{\mu\nu}$. This fact has relevant consequences in supergravity: in a gravity theory we know that the graviton is coupled to the conserved energy-momentum tensor, the presence of supersymmetry implies that the superpartner of the graviton, the gravitino, will be coupled to the conserved supercurrent S^μ which is in the same supermultiplet as $T^{\mu\nu}$.

(For more details see Weinberg III, section 26.7.)