Exercise 9.1 Ideal Quantum Gases in a Harmonic Trap

In this exercise we will discuss the difference of the bosonic respectively fermionic quantum corrections to a spinless ideal gas confined in a three-dimensional harmonic potential[1]. The energy states are given by

$$E_{\mathbf{a}} = \hbar\omega(a_x + a_y + a_z) \ , \tag{1}$$

where, as usually, we neglect the zero point energy of $E_0 = 3\,\hbar\omega/2$. Here, $\mathbf{a} = (a_x, a_y.a_z)$ denotes the occupation number of oscillator modes of the state $E_{\mathbf{a}}$ ($a_i = 0, 1, 2, ...$).

a) Derive the grand canonical partition function $\mathcal{Z}_{b,f}$ of this system and compute the grand potential $\Omega_{b,f}$ for bosons respectively fermions, i.e. show that

$$\Omega_f \propto f_4(z) \ , \qquad \Omega_b \propto g_4(z) \ , \tag{2}$$

where the functions $f_s(z)$ respectively $g_s(z)$ are defined as

$$f_s(z) = -\sum_{l=1}^{\infty} (-1)^l \frac{z^l}{l^s} \ , \qquad g_s(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^s} \ . \tag{3}$$

b) Derive the internal energy U, the specific heat C_V and the particle number N for the bosonic respectively fermionic system. Consider the high-temperature limit ($z \ll 1$). In order to relate U and C_V to N, introduce an expansion in a small parameter which depends on the particle number instead of the chemical potential. Define an effective volume $\mathcal{V}_{\text{eff}}$ related to the average square displacement of the harmonic oscillator $\langle r^2 \rangle$, i.e. $\mathcal{V}_{\text{eff}} = 4\pi/3\, \langle r^2 \rangle^{3/2}$. Define and compute the thermal expansion coefficient α using $\mathcal{V}_{\text{eff}}$.

c) Interpret your result of part b) by comparing the results to the corresponding results of the classical Boltzmann gas. In which way do quantum corrections influence bosonic respectively fermionic systems?

d) Find the critical temperature T_c at which Bose-Einstein condensation sets in.

Exercise 9.2 Bose-Einstein Condensation

a) In the lecture notes in sec. 4.5 an expression for the specific heat C_V of the spinless Bose gas has been derived for temperatures both above and below T_c (eq. (4.79)). As can be seen in fig. 4.4 in the same section, C_V does not diverge at T_c, but a cusp appears, suggesting that some T-derivative of C_V does diverge. Evaluate

$$\Delta = \lim_{T \to T_c^+} \partial_T\, C_V(T) - \lim_{T \to T_c^-} \partial_T\, C_V(T) \neq 0 \ , \tag{4}$$

to show that $\partial_T\, C_V(T)$ has a jump at T_c, and correspondingly $\partial_T^2\, C_V(T)$ diverges there.

b) As you have already seen in a), usually in the vicinity of a phase-transition several thermodynamic quantities show non-analytic behavior. Moreover, it turns out that the manner in which these quantities diverge can give useful information on the transition itself. Usually one finds a power-law behavior $X(T) \propto (T - T_c)^\gamma$ for some quantity X. γ is often called a

[1] For results of the classical ideal gas in a harmonic trap cf. section 2.4.3 of your lecture notes

critical exponent. Show that the compressibility of the Bose gas shows power-law behavior near T_c, and find the corresponding critical exponent!

Hint: At $T = T_c,\ z = 1$, so that the leading behavior of κ_T can be obtained by expanding in $\nu \equiv \ln z$ around $\nu = 0$.

c) Above we have considered the spinless Bose gas where thermodynamical quantities are related to the particle density and energy, but as for the fermions, bosons may have spin, so that magnetic properties become important. Adapt the calculation of the spin-susceptibility in ex. 8.2 (last exercise sheet) to the case of bosons with spin, and show that it diverges at $T = T_c$ in the limit $B \to 0$. Do you expect the system to be paramagnetic for $T < T_c$? If not, what do you expect? Try to give simple arguments for your conclusions!

Office Hours: Monday, November 16, 8-10 am (HIT K 12.2)