Solutions 7 - Mechanics

November 2, 2009

1. A Foretaste of General Relativity:

a) In spherical coordinates the Lagrangian is given by

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{M}{r} - \frac{\alpha}{r^3}, \tag{1}$$

With the Euler-Lagrange equation we obtain the equations of motion

$$\begin{aligned} l &= r^2\dot{\phi} = const, \\ \ddot{r} - \frac{l^2}{r^3} &+ \frac{M}{r^2} - 3\frac{\alpha}{r^4} = 0. \end{aligned} \tag{2}$$

Substituting $d\phi = lr^{-2}dt$ as well as $u = r^{-1}$ leads to the differential equation

$$u''(\phi) + u(\phi) = \frac{1}{l^2}\left(M - 3\alpha u^2(\phi)\right). \tag{3}$$

b) The unperturbed problem ($\alpha = 0$) has the solution

$$\begin{aligned} u_0(\phi) &= A_0 \sin\phi + A_1 \cos\phi + \frac{M}{l^2} \\ &= \frac{M}{l^2}\left(1 + \epsilon\cos\phi\right). \end{aligned} \tag{4}$$

Here we have used the boundary conditions $u_0(0) = ml^{-2}(1+\epsilon)$ and $u_0'(0) = 0$. We now insert (4) into the right-hand side of (3) to obtain the inhomogeneous, but linear differential equation

$$u'' + u = \frac{1}{l^2}\left[M - 3\alpha\left(\frac{M}{l^2}\right)^2\left(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi\right)\right] + O(\alpha^2). \tag{5}$$

A particular solution of (5) is given by $u_p = A_2 + \frac{1}{2}A_3\phi\sin\phi + \frac{1}{4}\epsilon A_3\left(1 - \frac{1}{3}\cos 2\phi\right)$, where $A_2 = Ml^{-2} - 3\alpha M^2 l^{-6}$ and $A_3 = -6\alpha\epsilon M^2 l^{-6}$. The general solution therefore is

$$u = A_0\sin\phi + A_1\cos\phi + A_2 + \frac{A_3}{2}\left(\phi\sin\phi + \frac{1}{2}\epsilon - \frac{1}{6}\epsilon\cos 2\phi\right) + O(\alpha^2), \tag{6}$$

We again use the boundary conditions $u'(0) = 0$ and $u(0) = Ml^{-2}(1+\epsilon)$ what leads to

$$u = \frac{M}{l^2}(1+\epsilon)\cos\phi + A_2(1-\cos\phi) + \frac{A_3}{2}\left[\phi\sin\phi + \frac{1}{2}\epsilon - \frac{1}{3}\epsilon\cos\phi - \frac{1}{6}\epsilon\cos 2\phi\right] + O(\alpha^2), \tag{7}$$

c) The perihelion is defined to be at $u' = 0$. Assuming a small shift $\Delta\phi$ we find by expanding

$$0 = u'(2\pi + \Delta\phi) = u'(2\pi) + u''(2\pi)\Delta\phi \qquad \Longrightarrow \qquad \Delta\phi \approx -\frac{u'(2\pi)}{u''(2\pi)}. \tag{8}$$

From equation (7) we get

$$\begin{aligned} u'(2\pi) &= -6\pi\alpha\epsilon M^2 l^{-6} + O(\alpha^2) \\ u''(2\pi) &= -\epsilon M l^{-2} + O(\alpha) \end{aligned} \tag{9}$$

and therefore

$$\Delta\phi = -\frac{6\pi\alpha M}{l^4} + O(\alpha^2) = -\frac{6\pi\alpha}{Ma^2(1-\epsilon^2)^2} + O(\alpha^2) \tag{10}$$

Here we have used $l = Ma(1-\epsilon^2)$.

2. **Amplitude of Oscillation**

The equation of the energy conservation reads

$$\frac{1}{2}m\dot{x}^2 + V(x) = E \qquad \Longrightarrow \qquad \frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - V(x)}. \tag{11}$$

Denoting the positions of the points of reversal by $x_1(E)$ and $x_2(E)$, we can write the periodic time as

$$T(E) = \sqrt{2m}\int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - V(x)}}. \tag{12}$$

After splitting the domain of integration, we can use V as an integration variable (note that $V(X_1) = V(x_2) = E$)

$$\begin{aligned} T(E) &= \sqrt{2m}\left\{\int_E^0 \frac{dV}{\sqrt{E-V}}\left.\frac{dx}{dV}\right|_{x<0} + \int_0^E \frac{dV}{\sqrt{E-V}}\left.\frac{dx}{dV}\right|_{x>0}\right\} \\ &= \sqrt{2m}\int_0^E \frac{dV}{\sqrt{E-V}}\left\{\frac{dx_2(V)}{V} - \frac{dx_1(V)}{V}\right\} \end{aligned} \tag{13}$$

Now we have found an expression for $T(E)$ that can be inserted into the expression given on the exercise sheet:

$$\int_0^E \frac{T(K)}{\sqrt{E-K}}dK = \sqrt{2m}\int_0^E dK\int_0^K dV\frac{1}{\sqrt{(E-K)(K-V)}}\left\{\frac{dx_2}{dV} - \frac{dx_1}{dV}\right\} \tag{14}$$

The triangular domain of integration can be re-parametrized by choosing V instead of E as the independent variable:

$$\int_0^E \frac{T(K)}{\sqrt{E-K}} dK = \sqrt{2m} \int_0^E dV \left\{ \frac{dx_2}{dV} - \frac{dx_1}{dV} \right\} \underbrace{\int_V^E dK \frac{1}{\sqrt{(E-K)(K-V)}}}_{\pi}$$
$$= \sqrt{2m}\pi \left[x_2(E) - x_1(E) \right] = \sqrt{2m}\pi d(E). \tag{15}$$

We therefore find the following expression for the peak-to-peak amplitude

$$d(E) = \frac{1}{\sqrt{2m}\pi} \int_0^E \frac{T(K)}{\sqrt{E-K}} dK. \tag{16}$$