Classical Mechanics
Solution Sheet 5

Exercise 5.1 Moving guiding curve : rotating pendulum

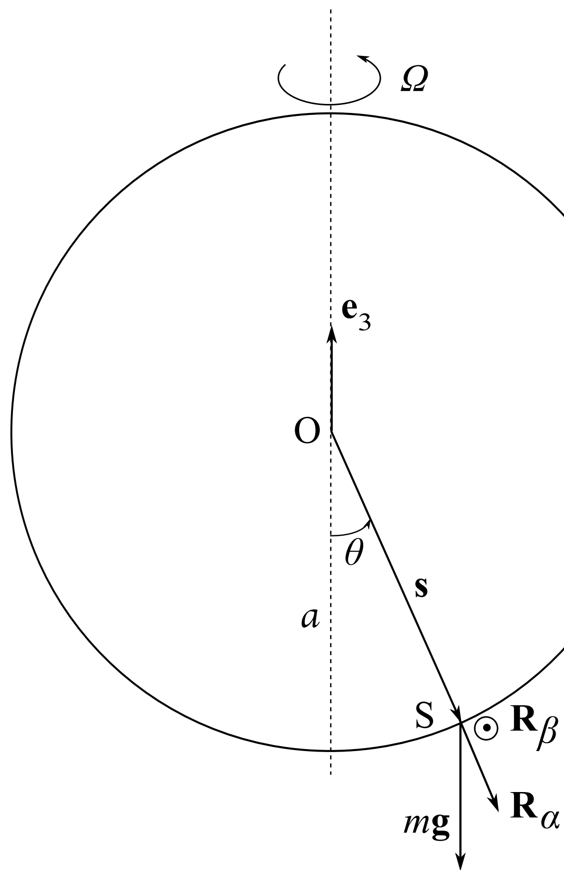

Abbildung 1:

Let us take a frame of reference of origin O that moves with the circle, see Fig. 1. The Poisson vector of this frame of reference is given by

$$\boldsymbol{\Omega} = \Omega \boldsymbol{e}_3$$

where $\boldsymbol{e}_3$ is the unitary vector along the vertical symmetry axis of the circle. The Newton's equation expressed in the rotating axis is given by

$$m\frac{\delta^2 \boldsymbol{s}}{\delta t^2} + 2m\boldsymbol{\Omega} \times \frac{\delta \boldsymbol{s}}{\delta t} + m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{s}) = m\boldsymbol{g} + \boldsymbol{R},$$

where the second term of the left hand side (Coriolis force) and the third term are fictitious forces due to the rotation of the circle, $\boldsymbol{R} = \boldsymbol{R}_\alpha + \boldsymbol{R}_\beta$ is the resultant of the reaction forces. Performing the scalar product between the latter equation and the relative velocity $\dfrac{\delta \boldsymbol{s}}{\delta t}$

which is always tangential to the curve, we get

$$\frac{\delta^2 \boldsymbol{s}}{\delta t^2} \cdot \frac{\delta \boldsymbol{s}}{\delta t} + (\boldsymbol{\Omega} \cdot \boldsymbol{s})(\boldsymbol{\Omega} \cdot \frac{\delta \boldsymbol{s}}{\delta t}) - \Omega^2 \boldsymbol{s} \cdot \frac{\delta \boldsymbol{s}}{\delta t} = \boldsymbol{g} \cdot \frac{\delta \boldsymbol{s}}{\delta t}.$$

After integration we have

$$\frac{1}{2} \left\| \frac{\delta \boldsymbol{s}}{\delta t} \right\|^2 + \frac{1}{2} (\boldsymbol{\Omega} \cdot \boldsymbol{s})^2 - \frac{1}{2} \Omega^2 \left\| \boldsymbol{s} \right\|^2 = \boldsymbol{g} \cdot \boldsymbol{s} + c, \tag{1}$$

where c is a constant.
We take now the polar coordinates of the rotating plane such that

$$r = a \ ; \qquad \boldsymbol{s} = a \boldsymbol{e}_r \ ; \qquad \frac{\delta \boldsymbol{s}}{\delta t} = a \dot{\theta} \boldsymbol{e}_\theta.$$

We introduce the following dimensionless variable

$$\tau = \omega t$$

and the dimensionless number

$$n = \frac{\Omega}{\omega} \qquad \text{where} \qquad \omega = \sqrt{\frac{g}{a}}$$

is the frequency of the simple pendulum with the same radius. We have

$$\begin{aligned} \boldsymbol{\Omega} \cdot \boldsymbol{s} &= -\Omega a \cos \theta \\ \boldsymbol{g} \cdot \boldsymbol{s} &= g a \cos \theta, \end{aligned}$$

and the equation (1) can be rewritten as

$$\frac{\dot{\theta}^2}{\omega^2} + \frac{\Omega^2}{\omega^2} \cos^2 \theta - \frac{\Omega^2}{\omega^2} = 2 \cos \theta + 2c$$

and since

$$\dot{\theta}^2 = \left(\frac{d\theta}{d\tau} \frac{d\tau}{dt} \right)^2 = \mathring{\theta}^2 \omega^2$$

where $\mathring{}$ is the derivative w.r.t τ, we finally have

$$\frac{\mathring{\theta}^2}{2} - (\frac{n^2}{2} \sin^2 \theta + \cos \theta) = c$$

where c is a constant fixed by the initial conditions.
In the latter equation the first term of the left hand side can be considered as a dimensionless kinetic energy $\mathcal{T}$, the second term as a dimensionless potential energy

$$\mathcal{V}(\theta) = -\frac{n^2}{2} \sin^2 \theta - \cos \theta,$$

and the right hand side as the dimensionless total energy of the system, which is a constant of the motion and fixed by initial conditions. Since the kinetic energy is definite positive we have

$$\mathcal{T} = c - \mathcal{V} \geq 0,$$

which means that the motion is possible only when the last inequality is satisfied, i.e. only for $c \geq \mathcal{V}(\theta)$.
In order to study the dynamics of this system we consider the function $\mathcal{V}(\theta)$ which derivatives are given by

$$\begin{aligned}
\mathcal{V}' &= (1 - n^2 \cos\theta)\sin\theta \\
\mathcal{V}'' &= n^2 + \cos\theta - 2n^2\cos^2\theta \\
\mathcal{V}''' &= (4n^2\cos\theta - 1)\sin\theta \\
\mathcal{V}'''' &= -4n^2 - \cos\theta + 8n^2\cos^2\theta.
\end{aligned}$$

We distinguish three cases :

1. $n < 1$ ($\omega > \Omega$: the gravitational force is dominant).
   We plot the function $\mathcal{V}(\theta)$ in Fig. 2 (for $n = 0.5$) which shows a minimum at $\theta = 0$ (stable relative equilibrium point) and a maximum at $\theta = \pi$ (unstable relative equilibrium point; "relative" because the stability is considered with respect to the moving frame of reference). The motion is possible only if $c \geq \min(\mathcal{V}) = -1$. In the Fig. 3 we plot the trajectories in the phase diagram in terms of the couples $(\theta, \mathring{\theta})$.

   - $c = -1$: The only solution is $\theta = 2k\pi$. The particle is at the bottom of the circle. This corresponds to the points $(2k\pi, 0)$ in the phase diagram.
   - $-1 < c < 1$: The particle is oscillating between two reflection angles $\pm\theta_r$. The trajectories on the phase diagram are closed describing a periodic orbit.
   - $c = 1$: If the initial angle is such that $\theta_0 \neq (2k+1)\pi$ the motion is asymptotically limited, the particle goes asymptotically to the top of the circle. The corresponding trajectory in the phase diagram is the ones crossing the θ-axis at $(2k+1)\pi$. If the initial angle is such that $\theta_0 = (2k+1)\pi$ the particle stays at the top of the circle.
   - $c > 1$: The motion of the particle is revolving. The trajectories do not cross the θ-axis of the phase diagram.

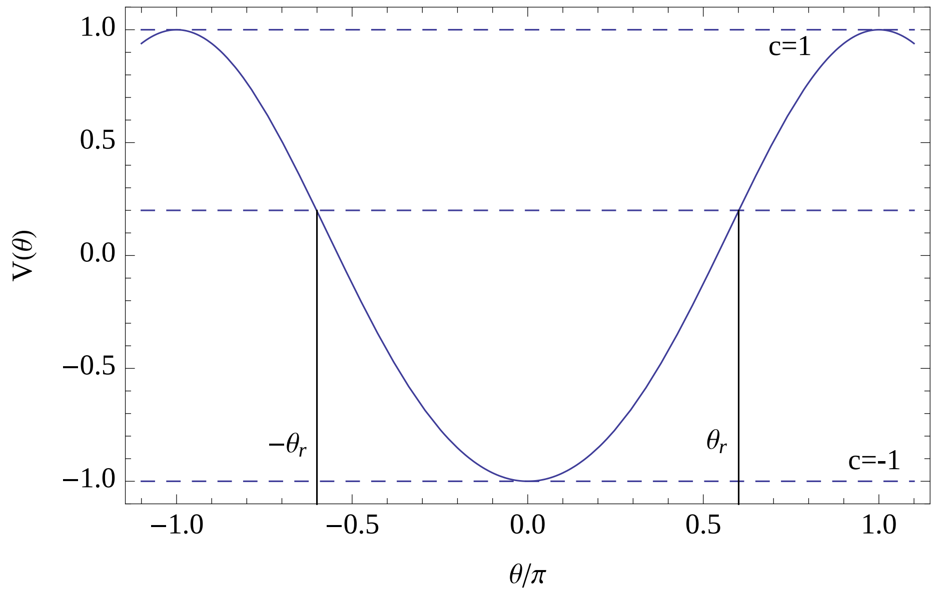

Abbildung 2: Dimensionless potential energy $\mathcal{V}(\theta)$ for $n = 0.5$.

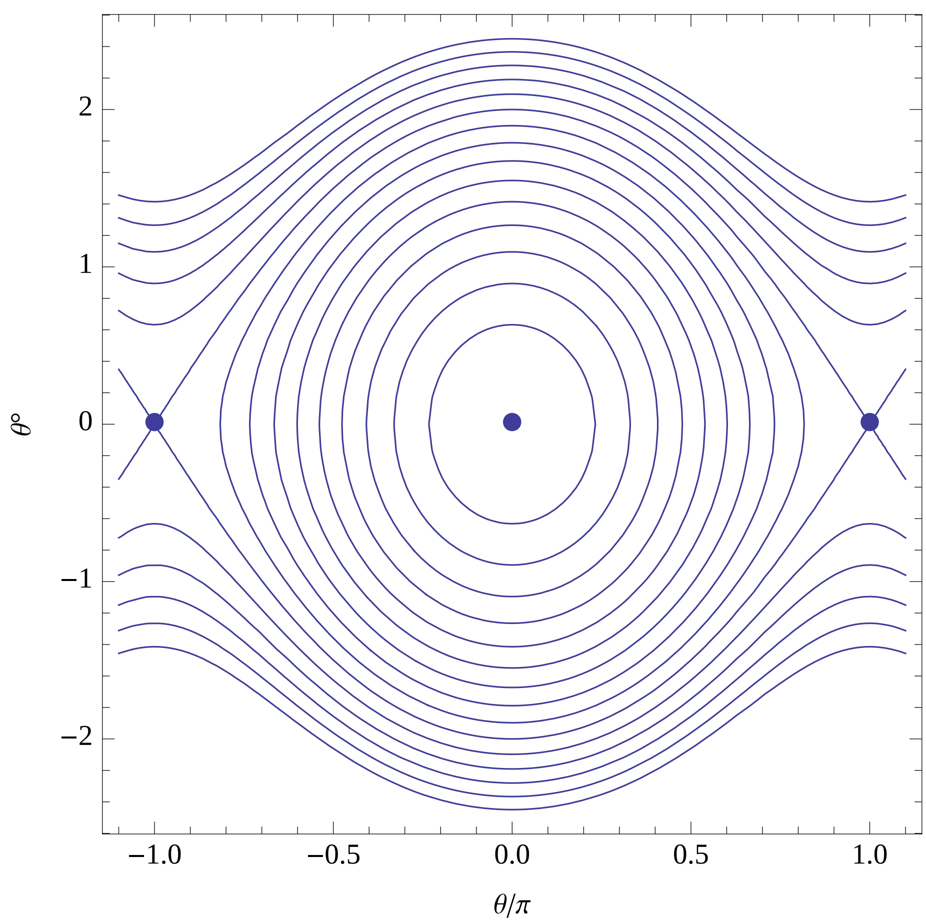

Abbildung 3: Phase diagram.

2. $n > 1$ ($\omega < \Omega$: the fictitious force due to the rotation of the circle is dominant). The function $\mathcal{V}(\theta)$ is plotted in Fig. (for $n = 1.5$). It shows two maxima at $\theta = 0$ and at $\theta = \pi$, both unstable relative equilibrium point, and one minimum at $\theta = \pm\theta_m$, a stable relative equilibrium point, given by $\cos\theta_m = 1/n^2$ (derived from the first derivative of $\mathcal{V}$). The phase diagram is plotted in Fig. 5.

   The motion is possible only for $c \geq \mathcal{V}(\theta_m) = -\dfrac{n^4+1}{2n^2}$.

   - $-(n^4+1)/2n^2 = c$: There are two relative equilibrium points at $\theta = \theta_m$ and $\theta = -\theta_m$ (stable). Those are the the two center located on the θ-axis (i.e. $\mathring{\theta} = 0$) in the phase diagram.
   - $-(n^4+1)/2n^2 < c < -1$: The particle oscillates between two reflection angles $0 < \theta_1 \leq \theta_m \leq \theta_2$ (or $-\theta_2 \leq -\theta_m \leq -\theta_1 < 0$). The trajectories in the phase diagram are closed describing a periodic orbit.
   - $c = -1$: If the initial angle is such that $\theta_0 \neq 2k\pi$ the motion is asymptotically limited, the particle goes asymptotically to the bottom of the circle. The corresponding trajectory in the phase diagram is crossing the θ-axis at $2k\pi$ (at the point $(0,0)$ in Fig. 5). If the initial angle is such that $\theta_0 = 2k\pi$ the particle stays at the bottom of the circle.
   - $-1 < c < 1$: The particle oscillates between two reflection angles $\pm\theta_r$ passing through the origin. The trajectories in the phase diagram are closed describing a periodic orbit.
   - $c = 1$: Similar to the previous section.
   - $c > 1$: Similar to the previous section.

3. $n = 1$
   The function $\mathcal{V}(\theta)$ has one maximum at $\theta = (2k+1)\pi$ and one minimum at $\theta = 2k\pi$. The discussion and the plots are similar to the case $n < 1$.

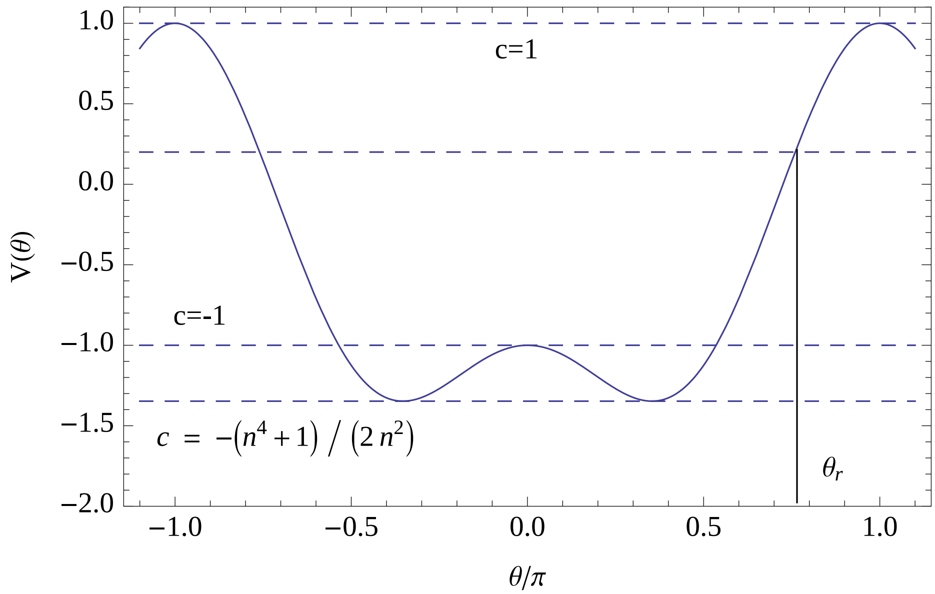

Abbildung 4: Dimensionless potential energy $\mathcal{V}(\theta)$ for $n = 1.5$.

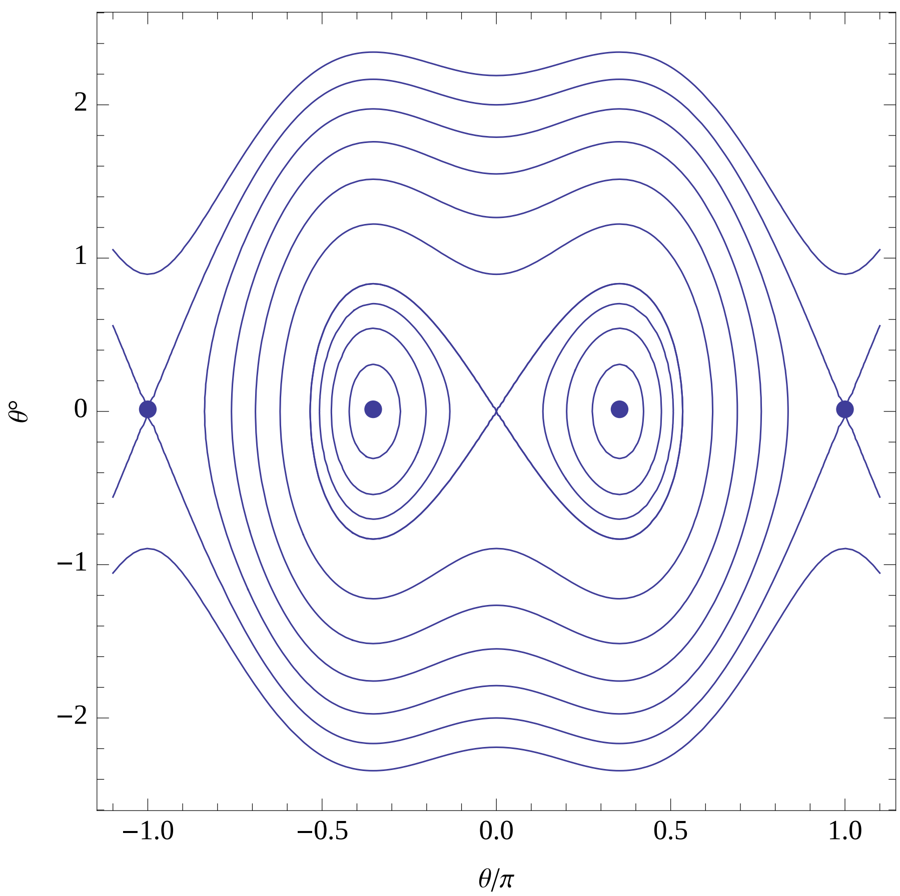

Abbildung 5: Phase diagram.

Exercise 5.2 Variable mass system

The differential equation of motion of the space shuttle is

$$m(t)\dot{\boldsymbol{s}} = m(t)\boldsymbol{g} + \mathcal{P} \ ,$$

where $\boldsymbol{s}$ is the position vector of the mass center of the shuttle and the thrust is given by

$$\mathcal{P} = \frac{dm}{dt}\boldsymbol{w} \ .$$

Since the space shuttle is starting at rest and that only vertical forces apply on it, the motion is following the vertical axis $\boldsymbol{E}_z$. Projecting the differential equation of motion on $\boldsymbol{E}_z$ and using $\boldsymbol{w} = -w\boldsymbol{E}_z$ we get

$$m(t)\ddot{z} = -m(t)g - \frac{dm}{dt}w \ ,$$

and

$$\ddot{z} = -g - \frac{w}{m}\frac{dm}{dt} \ .$$

In order to find the velocity of the shuttle at the end of the combustion we simply have to integrate this equation over the time from $t = 0$ to the time t_c. Since at t_c all the fuel is gone the total mass of the system is m_l, i.e.

$$\begin{array}{c|cc} t = 0 & \dot{z} = 0 & m = m_l + m_f \\ t = t_c & \dot{z} = \dot{z}_c & m = m_l \ . \end{array}$$

We then have

$$\int_0^{t_c} \ddot{z}dt = -\int_0^{t_c} gdt - \int_0^{t_c} \frac{w}{m}\frac{dm}{dt}dt \ ,$$

and

$$\dot{z}_c = -gt_c - \int_{m_l+m_f}^{m_l} w\frac{md}{m} = \ ,$$

and then

$$\dot{z}_c = -gt_c - w\ln\left(\frac{m_l}{m_l + m_f}\right) = -gt_c + w\ln\left(1 + \frac{m_f}{m_l}\right) \ .$$