Solutions 4

1. **Non-Uniqueness of the Lagrangian**

We show that adding a total time derivative of a function of the generalised coordinates to the Lagrangian L does not change the Euler-Lagrange equations, by direct computation. The q_i-Euler-Lagrange equation of L' is given by

$$\begin{aligned} 0 &= \partial_{q_i} L' - \frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{q}_i} L' = \partial_{q_i} L - \frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{q}_i} L + \partial_{q_i}\frac{\mathrm{d}}{\mathrm{d}t}F - \frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}F \\ &= \partial_{q_i} L - \frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{q}_i} L + \frac{\mathrm{d}}{\mathrm{d}t}\left[\partial_{q_i}F - \partial_{\dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}F\right]. \end{aligned}$$

Therefore L and L' lead to the same Euler-Lagrange equations iff

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\partial q_i F - \partial_{\dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}F\right] = 0.$$

Using the fact that F does not depend on q_i and that the total time derivative of F ist given by

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \partial_t F + \sum_{j=1}^{n} \dot{q}_j \partial_{q_j} F,$$

we see that

$$\partial_{\dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}F = \partial_{q_i}F.$$

This proves the claim.

2. **Point Particle Gliding in a Cone**

We use spherical coordinates and the position $\vec{x}$ of the particle is given by

$$\vec{x} = r\begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix},$$

where r is the distance of the particle from the apex and φ is an angular variable around the axis of the cone. Note that r and φ are the generalised coordinates of the system, but θ isn't, because θ describes the constant aperture of the cone.

a) To determine the kinetic energy we need the velocity of the particle

$$\dot{\vec{x}} = \dot{r} \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} + r\dot{\varphi}\sin\theta \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}.$$

The kinetic energy T and potential energy V are then given by

$$T = \frac{m}{2}\dot{\vec{x}}^2 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 \sin^2\theta) \qquad\qquad U = mgr\cos\theta,$$

resulting in the Lagrangian

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 \sin^2\theta) - mgr\cos\theta.$$

b) We first compute the Euler-Lagrange equation for φ.

$$\partial_\varphi L = 0 \qquad\qquad \partial_{\dot{\varphi}} L = mr^2\dot{\varphi}\sin^2\theta$$

The Euler-Lagrange equation for φ is therefore given by

$$\frac{\mathrm{d}}{\mathrm{d}t} mr^2\dot{\varphi}\sin^2\theta = 0,$$

or

$$mr^2\dot{\varphi}\sin^2\theta = P_\varphi,$$

where P_φ is a constant. Next we compute the Euler-Lagrange equation for r.

$$\partial_r L = mr\dot{\varphi}^2\sin^2\theta - mg\cos\theta \qquad\qquad \partial_{\dot{r}} L = m\dot{r}.$$

Cancelling the common factors of m and solving for $\ddot{r}$ the Euler-Lagrange equation for r reads

$$\ddot{r} = r\dot{\varphi}^2\sin^2\theta - g\cos\theta.$$

c) The component L_3 of the angular momentum $\vec{x} \times m\dot{\vec{x}}$ along the axis of the cone is $mr^2\dot{\varphi}\sin^2\theta$. This is just the left hand side of the Euler-Lagrange equation for φ. Therefore $L_3 = P_\varphi$ is constant.

d) To determine the solutions with constant r we must first eliminate $\dot{\varphi}$ from the Euler-Lagrange equation for r. We do this by first isolating $\dot{\varphi}$ in the Euler-Lagrange equation for φ

$$\dot{\varphi} = \frac{P_\varphi}{mr^2\sin^2\theta}$$

and plugging the result into the Euler-Lagrange equation for r

$$\ddot{r} = \frac{P_\varphi^2}{m^2 r^3 \sin^2\theta} - g\cos\theta.$$

For solutions with constant r we have $\ddot{r} = 0$ and therefore

$$r = \left(\frac{P_\varphi^2}{m^2 g\cos\theta\sin^2\theta}\right)^{1/3}$$

3. **Nut winding down a thread**

We will refer to the hight of the centre of mass of the nut by z and how far it has been rotated by φ. Since z can be expressed as a function of φ

$$z(\varphi) = -\frac{\varphi h}{2\pi},$$

φ is our only generalised coordinate.

a) To determine the Lagrangian we need the moment of inertia

$$\begin{aligned}
I &= \rho\int_0^\ell \int_0^{2\pi} \int_r^R \tilde{r}^2 \mathrm{d}\tilde{r} r \mathrm{d}\varphi \mathrm{d}z \\
&= 2\pi\ell\rho \left.\frac{\tilde{r}^4}{4}\right|_r^R = 2\pi\ell\rho\frac{R^4 - r^4}{4} \\
&= 2\pi\ell\rho\frac{R^2 - r^2}{4}(R^2 + r^2) = \frac{m}{2}(R^2 + r^2),
\end{aligned}$$

where $m = \pi\ell\rho(R^2 - r^2)$ is the total mass of the nut. The kinetic and potential energies are then given by

$$\begin{aligned}
T &= \frac{m}{2}\frac{\dot{\varphi}^2 h^2}{(2\pi)^2} + \frac{m}{4}(R^2 + r^2)\dot{\varphi}^2 \\
V &= -mg\frac{h\varphi}{2\pi}
\end{aligned}$$

and the Lagrangian by

$$L = \frac{m}{2}\frac{\dot{\varphi}^2 h^2}{(2\pi)^2} + \frac{m}{4}(R^2 + r^2)\dot{\varphi}^2 + mg\frac{h\varphi}{2\pi}.$$

b) The φ and $\dot{\varphi}$ derivatives are

$$\partial_\varphi L = mg\frac{h}{2\pi} \qquad \begin{aligned}
\partial_{\dot{\varphi}} L &= m\frac{\dot{\varphi} h^2}{(2\pi)^2} + \frac{m}{2}(R^2 + r^2)\dot{\varphi} \\
&= \left(\frac{h^2}{(2\pi)^2} + \frac{R^2 + r^2}{2}\right) m\dot{\varphi}.
\end{aligned}$$

Canceling the common factor m the Euler-Lagrange equation is therefore

$$\ddot{\varphi} = \frac{g}{\frac{h}{2\pi} + (R^2 + r^2)\frac{\pi}{h}}.$$

Assuming $\varphi(0) = \dot{\varphi}(0) = 0$ the Euler-Lagrange equation integrates to

$$\varphi(t) = \frac{gt^2}{\frac{h}{\pi} + 2(R^2 + r^2)\frac{\pi}{h}},$$

or in terms of the vertical coordinate z

$$z(t) = -\frac{1}{2}\frac{ght^2}{h + 2(R^2 + r^2)\frac{\pi^2}{h}}.$$

The nut accelerates at a factor

$$\frac{h}{h + 2(R^2 + r^2)\frac{\pi^2}{h}} \leq 1$$

times the acceleration it would have when falling freely.