Solutions 3

October 6, 2009

1. Throwing up a stone

Answer:

We consider the problem in the inertial system with the earth's centre as origin.The trajectory of the thrown body will lie on the plan $span(\vec{v}_i, \vec{r}_i)(i \equiv$initial), as usual for a 2-bodies system.
Since the gravitational force has form $-\frac{k}{r^2}\vec{e}_r$ the Lenz vector

$$\vec{A} := \vec{p} \times \vec{L} - k.m\vec{e}_r \tag{1}$$

is conserved. Remember that this vector points in the direction of the perihel of the trajectory.

$$\begin{aligned} \vec{A} &= m \cdot (\vec{v}_0 + \vec{v}_R) \times (\vec{r} \times (m \cdot (\vec{v}_0 + \vec{v}_R))) - GMm^2\vec{e}_r = \\ &= m \cdot (v_0\vec{e}_r + v_R\vec{e}_\phi) \times (r\vec{e}_r \times (m \cdot (v_0\vec{e}_r + v_R\vec{e}_\phi))) - GMm^2\vec{e}_r = \\ &= m^2 r v_R v_0(-\vec{e}_\phi) + m^2 r v_R^2 \vec{e}_r - GMm^2\vec{e}_r \end{aligned} \tag{2}$$

where $\vec{v}_R$ is the component of the initial velocity of the body due to the earth's rotation.
The angle between the 2 points in wich the ellipse of the body intersects the earth's surface will be called Θ. The angle between start point and Lenz vector is θ.(Note:$\Theta = 2(\pi - \theta)$, since the top the the trajectory of the thrown body corresponds to the aphelion of the ellipse).

$$cos\theta = \frac{\vec{A} \cdot \vec{e}_r}{|\vec{A}|} = \frac{v_R^2 - GM/R}{\sqrt{(v_R v_0)^2 + (v_R^2 - GM/R)^2}} \tag{3}$$

($R =$ earth's radius) or with $v_R = R\omega cos(\phi)$

$$cos(\theta) = \frac{R\omega^2 cos(\phi)^2 - g}{\sqrt{\omega^2 cos(\phi)^2 v_0^2 + (R\omega^2 cos(\phi)^2 - g)^2}} \approx -1 + \frac{1}{2}\left(\frac{\omega cos(\phi) v_0}{g - R\omega^2 cos(\phi)^2}\right)^2 \tag{4}$$

$$\Longrightarrow -1 + \frac{1}{2}(\pi - \theta)^2 \approx -1 + \left(\frac{v_0 \omega cos(\phi)}{g}\right)^2 \tag{5}$$

$$\Longrightarrow \Delta x_s = R\Theta \approx \frac{2v_0 R\omega cos(\phi)}{g} \tag{6}$$

$(g = \frac{GM}{R^2})$.We assumed $\omega << 1$, and we worked in 1.order. While the body has accomplished his trajectory, our position changed (in 1. order in $v_0\omega$ we have $\Delta t\omega \approx \frac{2\omega v_0}{g}$).

$$\Delta x_{obs} = Rcos(\phi)\omega\Delta t \tag{7}$$

We calculate $\Delta t(\omega, v_0)$ in 0.order in ω and 1.order in v_0^2 (or equiv. 1.order in $(r-R)$),i.e

$$F_{grav}(h := (r-R)) \approx -g + 2\frac{g}{R}h \tag{8}$$

$$h'' = -g + 2\frac{g}{R}h \tag{9}$$

$$\Longrightarrow h_0 = -\frac{g}{2}t^2 + v_0 t \tag{10}$$

$$\Longrightarrow h_1 = (-\frac{g}{2}t^2 + v_0 t) + \frac{g}{R}(-\frac{g}{12}t^4 + \frac{v_0}{3}t^3) \tag{11}$$

From

$$h_1(\Delta t) = 0 \tag{12}$$

It follows $(\Delta t = 2v_0/g + \epsilon)$ by considering just terms linear in ϵ

$$\epsilon \approx \frac{4v_0^3}{3g^2 R} \tag{13}$$

$$\Longrightarrow \Delta x := \Delta x_{obs} - \Delta x_s \approx \frac{4v^3\omega cos(\phi)}{3g^2} \tag{14}$$

Äquivalent solution We have the potential field $U = -m\mathbf{g}\mathbf{r}$ where $\mathbf{g}$ is the acceleration. The rotation of the earth is uniform so that $\dot{\Omega} = 0$. Neglecting the centrifugal force we have the equation

$$\dot{\mathbf{v}} = 2\mathbf{v} \times \mathbf{\Omega} + \mathbf{g} \tag{15}$$

We solve the equation through successive approximation. Take $\mathbf{v} = \mathbf{v_1} + \mathbf{v_2}$ where $\dot{\mathbf{v_1}} = \mathbf{g}$ and consequently $\mathbf{v_1} = \mathbf{g}t + \mathbf{v_0}$. We insert in the right hand side of equation 15 only $\mathbf{v_1}$ so the equation simplifies to

$$\dot{\mathbf{v}} = 2\mathbf{v_1} \times \Omega + \mathbf{g} \tag{16}$$

$$\Longrightarrow \dot{\mathbf{v_2}} = 2t\mathbf{g} \times \Omega + 2\,\mathbf{v_0} \times \Omega \tag{17}$$

And we have the equation for $\mathbf{v_2}$. After integration of $\dot{\mathbf{v}} = \dot{\mathbf{v_1}} + \dot{\mathbf{v_2}}$ using the equation 17 and $\dot{\mathbf{v_1}} = \mathbf{g}$ we get

$$\mathbf{r} = \frac{t^3}{3}\mathbf{g} \times \Omega + t^2\mathbf{v_0} \times \Omega + \frac{t^2}{2}\mathbf{g} + t\mathbf{v_0} + \mathbf{r_0} \tag{18}$$

We choose the coordinate system with z-axis pointing upwards and the x-axis pointing from the meridian zero to the north pole. Then we have $g_x = g_y = 0$ and $g_z = -g$ as well as $\Omega_x = \Omega \cos\lambda$,$\Omega_y = 0$ and $\Omega_z = \Omega \sin\lambda$, where λ is the latitude. As initial conditions we have $\mathbf{r_0} = 0$ and $\mathbf{v_0} = v_z \mathbf{e_z}$, inserting them in 18 we get

$$r_y = -\frac{1}{3}t^3 g\Omega\cos\lambda + v_z t^2 \Omega \cos\lambda \tag{19}$$

$$r_z = -\frac{1}{2}gt^2 + v_z t \tag{20}$$

For the flying time of the stone you get $t = \frac{2v_z}{g}$ by equating $r_z = 0$ and inserting this in 19 we get the horizontal diplacement in $\mathbf{e_y}$ direction of the stone

$$y = -\frac{8v_z^3}{3g^2}\Omega\cos\lambda + 4\frac{v_z^3}{g^2}\Omega\cos\lambda = \frac{4v_z^3}{3g^2}\Omega\cos\lambda \tag{21}$$

2. **Euler-Lagrange**

Answer:

1. The moment of inertia is defined as

$$I = \int_V r^2 \rho(\vec{r}) dV, \tag{22}$$

where r is the distance to the rotation axis and ρ is the density. For a homogeneous cylinder of radius r and length l we have in cylinder coordinates

$$\begin{aligned} I &= \int_0^l \int_0^{2\pi} \int_0^r r'^2 \rho\, r' dr' d\phi\, dl' \\ &= \rho l \pi r^2 \frac{r^2}{2} \\ &= m\frac{r^2}{2}, \end{aligned} \tag{23}$$

where we used the formula

$$m = \rho l \pi r^2 \tag{24}$$

for the mass.

2. The Lagrangian is given by

$$L = T - V. \tag{25}$$

There are 3 contributions to the kinetic energy T. The angular momentum of m_1 and m_2 and the linear momentum of m_2. Using

$$z = \varphi_1 r_1 + \varphi_2 r_2, \tag{26}$$

we find

$$T = \frac{I_1}{2}\dot{\varphi}_1^2 + \frac{I_2}{2}\dot{\varphi}_2^2 + \frac{m_2}{2}\left(\dot{\varphi}_1 r_1 + \dot{\varphi}_2 r_2\right)^2. \tag{27}$$

The gravitation potential is given by

$$V = -m_2 g \left(\varphi_1 r_1 + \varphi_2 r_2\right). \tag{28}$$

Thus

$$L = \frac{m_1 r_1^2}{4}\dot{\varphi}_1^2 + \frac{m_2 r_2^2}{4}\dot{\varphi}_2^2 + \frac{m_2}{2}\left(\dot{\varphi}_1 r_1 + \dot{\varphi}_2 r_2\right)^2 + m_2 g \left(\varphi_1 r_1 + \varphi_2 r_2\right). \tag{29}$$

3. Inserting the Lagrangian in the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_i}\right) - \frac{\partial L}{\partial \dot{\varphi}_i} = 0, \tag{30}$$

we find

$$\left(\frac{m_1}{2} + m_2\right) r_1 \ddot{\varphi}_1 + m_2 r_2 \ddot{\varphi}_2 - m_2 g = 0 \tag{31}$$

$$\frac{3}{2} m_2 r_2 \ddot{\varphi}_2 + m_2 r_1 \ddot{\varphi}_1 - m_2 g = 0. \tag{32}$$

Finally

$$\ddot{\varphi}_1 = \frac{2 m_2 g}{(3m_1 + 2m_2) r_1} \tag{33}$$

$$\ddot{\varphi}_2 = \frac{2 m_1 g}{(3m_1 + 2m_2) r_2}. \tag{34}$$

4. The time evolution of φ_i is given by

$$\varphi_i(t) = \frac{a_i}{2} t^2, \tag{35}$$

where a_i is the constant acceleration given in the equations of motion. Substituting for φ_i in the expression of z we find

$$\begin{aligned} z(t) &= (r_1 \varphi_1 + r_2 \varphi_2) \\ &= g \frac{m_1 + m_2}{3m_1 + 2m_2} t^2. \end{aligned} \tag{36}$$