Solutions 1 - reminder of Newtonian mechanics

September 21, 2009

1. **Geostationary space station**

a) The observer must be on the equator of the earth. The orbit of the space station is a large circle in the equatorial plane with center at the center of the earth.

b) Since the centripetal force is equal to the gravitational force we have

$$\frac{mv^2}{R} = \frac{GM_\oplus m}{R^2}, \tag{1}$$

where $M_\oplus$ is the mass of the earth, v is the circular velocity and R the orbital radius of the space station. With

$$v = \frac{s}{t} = \frac{2\pi R}{T} \tag{2}$$

we find the following distance L between the observer and the space station:

$$L = R - R_\oplus = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} - R_\oplus. \tag{3}$$

with $T = 24h$, $M_\oplus = 6.0 \cdot 10^{25} kg$ and $R_\oplus = 6.4 \cdot 10^6 m$ we finally obtain

$$L = 3.6 \cdot 10^4 km. \tag{4}$$

2. **Circling particle**

a) The tension in the string provides the centripetal force needed for the circular motion, hence $F = mv_0^2/R_0$.

b) The angular momentum of the mass m is $J = mv_0 R_0$, the kinetic energy is $T = mv_0^2/2$.

c) The radius of the circular motion of the mass m decreases when the tension in the string is increased gradually. The angular momentum is conserved, thus

$$mv_0R_0 = mv_1(\frac{R_0}{2}) \qquad \Longrightarrow \qquad v_1 = 2v_0. \tag{5}$$

The final kinetic energy is then

$$T_1 = \frac{1}{2}mv_1^2 = 2mv_0^2. \tag{6}$$

The reason why the pulling of the string should be gradual is that the vectors $\mathbf{r}$ and $\mathbf{v}$ must stay approximately perpendicular.

3. **Fast rotating planet**

The conservation of energy can be written as the statement

$$E = T(R) + U(R) = T(\infty) + U(\infty) = const. \tag{7}$$

The escape velocity v_e can now be determined by setting $T(\infty)$ to zero. This leads to

$$\begin{aligned} \frac{1}{2}mv_e^2(R) &= \int_R^\infty F(r)dr = GMm\int_R^\infty \frac{1}{r^2}dr = \frac{GMm}{R}, \\ v_e(R) &= \sqrt{\frac{2GM}{R}}. \end{aligned} \tag{8}$$

On our fast rotation planet we have $g_{pol} = 2g_{eq}$, as well as

$$g_{pol} = \frac{GM}{R^2} \qquad\qquad g_{eq} = \frac{GM}{R^2} - \frac{v^2}{R}, \tag{9}$$

what leads to the equation

$$\frac{GM}{R} = 2v^2. \tag{10}$$

Substituting eq. (10) into eq. (8) gives the final result

$$v_e = 2v. \tag{11}$$

3. **Sphere**

a) Conservation of energy gives (see Fig. 1)

$$\begin{aligned} E = mg(2R) &= \frac{1}{2}mv^2 + mgR(1+\cos\theta), \\ \frac{1}{2}mv^2 &= mgR(1-\cos\theta). \end{aligned} \tag{12}$$

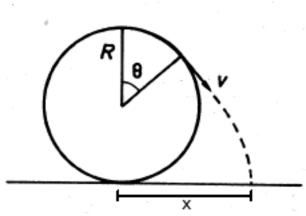

Figure 1: Particle falling from a sphere.

The radial force the sphere exerts on the particle is

$$F = mg\cos\theta - \frac{mv^2}{R}. \tag{13}$$

When $F = 0$, the constraint vanishes and the particle leaves the sphere. At this instance we have

$$v^2 = gR\cos\theta \qquad v^2 = 2gR(1-\cos\theta) \tag{14}$$

giving

$$\cos\theta = 2/3, \qquad v = \sqrt{\frac{2gR}{3}}. \tag{15}$$

b) After leaving the sphere, the particle follows a parabolic trajectory until it hits the plane. The trajectory is described by

$$\begin{aligned} x(t) &= vt\cos\theta + x_0 \\ y(t) &= -\frac{1}{2}gt^2 - vt\sin\theta + y_0 \end{aligned} \tag{16}$$

with the initial conditions $x_0 = R\sin\theta$ and $y_0 = R(1+\cos\theta)$. This can also be written as

$$y(x) = y_0 - \tan\theta(x-x_0) - \frac{g}{2v^2\cos^2\theta}(x-x_0)^2 \tag{17}$$

Setting $y(x) = 0$ and solving for x leads to

$$x - x_0 = \frac{v}{g}\cos\theta\left(-v\sin\theta \pm \sqrt{v^2\sin^2\theta + \frac{1}{2}gy_0}\right). \tag{18}$$

Substituting θ and v and only considering the positive solution leads to

$$x = \frac{\sqrt{5}}{27}(5+2\sqrt{13})R \simeq 1.01R. \tag{19}$$

The result is independent of g!

5. **Enjoying wine in a train**

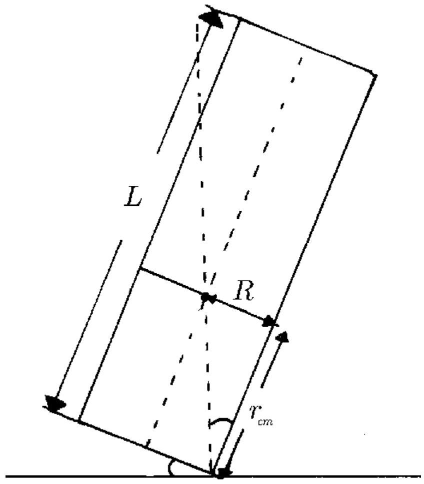

Figure 2: Illustration of the the bottle.

Let us study a bottle that has been tilted as in Fig. 2. To simplify calculations, we ignore the fact that the surface of the wine is always horizontal. When the center of mass r_{cm} lies on the y-axis, there is no net torque. Let us call the angle that gives us this configuration θ_{eq}. For $\theta < \theta_{\mathrm{eq}}$, torque will force the bottle back into a standing position and for $\theta > \theta_{\mathrm{eq}}$ the bottle will fall. In essence, the larger θ_{eq} is, the more stable is the bottle!

The setup is shown in Fig. 2. The equilibrium angle is calculated through

$$R/r_{\mathrm{cm}} = \tan(\theta_{\mathrm{eq}}), \tag{20}$$

where R is the radius of the bottle. Maximizing θ_{eq} is equivalent to minimizing r_{cm} which is given by

$$r_{\mathrm{cm}} = \frac{m_{\mathrm{b}}\frac{L}{2} + m_{\mathrm{w}}x\frac{L}{2}}{m_{\mathrm{b}} + m_{\mathrm{w}}}, \tag{21}$$

where m_{b} is the mass of an empty wine bottle, m_{w} is mass of the wine and x is the fraction of wine in the bottle. By using the fact that the wine mass is $m_{\mathrm{w}} = x\, m_{\mathrm{w,f}}$, where $m_{\mathrm{w,f}}$ is the mass of wine in a full bottle we can expand into

$$r_{\mathrm{cm}} = \frac{m_{\mathrm{b}}\frac{L}{2} + m_{\mathrm{w,f}}\frac{L}{2}x^2}{m_{\mathrm{b}} + m_{\mathrm{w,f}}x}. \tag{22}$$

The sought solution is when $\frac{dr_{\mathrm{cm}}}{dx} = 0$ and $\frac{d^2r_{\mathrm{cm}}}{dx^2} > 0$ (minimum of function).

The derivative is

$$\frac{dr_{\rm cm}}{dx} = \frac{(m_{\rm b} + m_{\rm w,f}x)m_{\rm w,f}Lx - m_{\rm w,f}(m_{\rm b}\frac{L}{2} + m_{\rm w,f}\frac{L}{2}x^2)}{(m_{\rm b} + m_{\rm w,f}x)^2} \tag{23}$$

Equating this to zero and simplifying renders the quadratic equation

$$x^2 + 2\frac{m_{\rm b}}{m_{\rm w,f}}x - \frac{m_{\rm b}}{m_{\rm w,f}} = 0 \tag{24}$$

which has the solution

$$x = -\frac{m_{\rm b}}{m_{\rm w,f}} \pm \sqrt{\left(\frac{m_{\rm b}}{m_{\rm w,f}}\right)^2 + \frac{m_{\rm b}}{m_{\rm w,f}}}. \tag{25}$$

We see that the optimum amount of wine in the bottle only depends on the mass of the glass bottle itself and the mass of the wine itself *in a full bottle*. By using $m_{\rm b} = 450g$ and $m_{\rm w,f} = 750g$ we get $x = 0.3798 \approx 3/8$. For a standard bottle of wine, $R \approx L/6$. Using this, and the solution for x, in Eq. 20 we see that the maximum angle of stability is $\theta_{\rm eq} \approx 23.7$.