Statistical Physics
Exercise Sheet 5

Exercise 5.1 Drude Conductivity

The goal of this exercise is to extend the calculation of the conductivity of an electron gas in the relaxation time approximation (as discussed in the lectures) to the case of a time-dependent electric field, $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$.

a) Splitting the distribution function into the equilibrium distribution $f_0(p)$and a small perturbation $g(p,t)$,

$$f(p,t) = f_0(p) + g(p,t), \tag{1}$$

find expressions for $g(p,t)$ and σ_ω (defined by $\vec{j}_\omega = \sigma_\omega \vec{E}$, $\vec{E}$ as above), using approximations similar to those in the lecture notes. Calculate the time-dependent current $\vec{j}(t)$ for an external field given by $\vec{E}(t) = \vec{E}_0 \cos\omega t$. Show that in the limiting case $\omega \to 0$ the DC conductivity is recovered.

b) From the result of a), calculate

$$p = \langle \vec{j} \cdot \vec{E} \rangle_t = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} dt \vec{j}(t) \cdot \vec{E}(t). \tag{2}$$

This term describes Ohmic heating (it is the local version of $P = UI$), which is accompanied by an increase in entropy. Explain why there is no increase in entropy found when using $\dot{S} = -\int dp^3 \dot{f}(p) \left(1 + \log f(p)\right)$. What are the implicit assumptions made in the ansatz (1), and where does the entropy production take place if these assumptions are justified?

c) Clearly, the derivation of the Drude conductivity in a) used some drastic approximations. Unfortunately, for most interesting systems, this is the usual state of affairs, rather than an exception. However, for the so-called response functions, of which the conductivity is an example, there exist exact identities, the Kramers-Kronig-relations

$$\text{Re } \sigma_\omega = \frac{1}{\pi} \text{P} \int_\infty^\infty d\nu \frac{\text{Im } \sigma_\nu}{\nu - \omega}, \tag{3}$$

$$\text{Im } \sigma_\omega = \frac{-1}{\pi} \text{P} \int_\infty^\infty d\nu \frac{\text{Re } \sigma_\nu}{\nu - \omega}, \tag{4}$$

which can be used to check the plausibility of an approximation. As these relations are a consequence of causality, any reasonable approximation of the conductivity should fulfill them. Verify that your result obtained in part a) satisfies the Kramers-Kronig relations!

Exercise 5.2 Maxwell-Boltzmann distribution function for relativistic particles

Find the equilibrium distribution function for relativistic particles of energy $E(\vec{p}) = \sqrt{p^2c^2 + m^2c^4}$, where m is the mass and c the speed of light. Show that in the limit $k_B T \ll mc^2$ the usual Maxwell-Boltzmann distribution function is recovered. Calculate the internal energy U and the specific heat C_V and find the first relativistic corrections to these expressions.

Hints: The integral representation of the modified Bessel functions of the second kind is

$$K_n(z) = \int_0^\infty e^{-z \cosh y} \cosh(ny) dy. \tag{5}$$

The asymptotic behavior for $z \to \infty$ is

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4n^2 - 1}{8z} + \frac{(4n^2-1)(4n^2-9)}{2!(8z)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8z)^3} + \dots \right]. \tag{6}$$