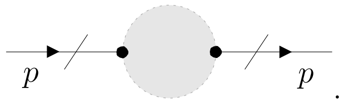
Quantum Field Theory I, Exercise Set 11.

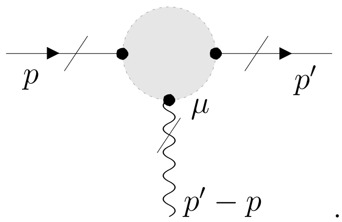
HS 08 *Due:* 11/12 December 2008

1. Ward identity

The electron self-energy $-\mathrm{i}\Sigma(p)$ is the sum of all amplitudes of amputated 1PI diagrams contributing to the electron propagator. Graphically we denote this sum by

$-\mathrm{i}\Sigma(p) =$ 

Denote with $-\mathrm{i}e_{\mathrm{R}}\Lambda_\mu(p',\, p)$ the three-point vertex function. Graphically this is

$-\mathrm{i}e_{\mathrm{R}}\Lambda_\mu(p',\, p) =$ 

Splitting off the first-order contribution we may write $-\mathrm{i}e_{\mathrm{R}}\Lambda_\mu(p',\, p) = -\mathrm{i}e_{\mathrm{R}}\gamma_\mu - \mathrm{i}e_{\mathrm{R}}\Gamma_\mu(p',\, p)$. If the theory is renormalised in a gauge invariant way, the Ward identity states

$$-\frac{\partial}{\partial p^\mu}\Sigma(p) = \Gamma_\mu(p,\, p)\,.$$

For definiteness we regularise the theory according to Pauli-Villars and impose a cut-off on the internal photon momenta. For simplicity we work in the Feynman gauge. Prove the Ward identity to one-loop order by explicit calculation.

2. Finiteness of light-by-light scattering

We consider light-by-light scattering, i.e. $\gamma\gamma \to \gamma\gamma$, at one-loop order in QED.

(i) Draw all 6 Feynman diagrams in momentum space contributing to the one-loop amplitude $\Pi(k_1, k_2, k_3, k_4)$.

(ii) We write $\Pi(k_1, k_2, k_3, k_4) = \varepsilon^\mu(k_1)\varepsilon^\nu(k_2)\varepsilon^\rho(k_3)\varepsilon^\sigma(k_4)\Pi_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4)$. Convince yourself that on the one-loop level this is of the form

$$\Pi_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) = T_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) + T_{\mu\nu\sigma\rho}(k_1, k_2, k_4, k_3) + T_{\mu\rho\nu\sigma}(k_1, k_3, k_2, k_4)\,.$$

(iii) We are only interested in the behaviour of the amplitude for large p. Therefore we assume $|p| \gg |k_i|$ for $i = 1, 2, 3, 4$ and $|p| \gg m$. Show that the potentially divergent term in $\Pi_{\mu\nu\rho\sigma}$ is of the form

$$A \int \mathrm{d}^4 p\, \frac{(p^2)^2}{\left(p^2\right)^4}$$

with

$$
\begin{aligned}
A = &\left(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}\right) \\
\times &\left[\mathrm{Tr}\{\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\rho\gamma^\gamma\gamma_\sigma\gamma^\delta\} + \mathrm{Tr}\{\gamma_\mu\gamma^\alpha\gamma_\rho\gamma^\beta\gamma_\nu\gamma^\gamma\gamma_\sigma\gamma^\delta\} + \mathrm{Tr}\{\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\sigma\gamma^\gamma\gamma_\rho\gamma^\delta\}\right] .
\end{aligned}
\tag{1}
$$

Hint: By symmetry, we have

$$
\int \mathrm{d}^4p \, \frac{p_\alpha p_\beta p_\gamma p_\delta}{\left(p^2\right)^4} = \left(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\delta}\right) \int \mathrm{d}^4p \, \frac{(p^2)^2}{\left(p^2\right)^4} .
$$

(iv) Compute A defined in (1). What is the conclusion concerning the finiteness of the amplitude?

Hints:

- $\gamma^\mu\gamma^\nu\gamma_\mu = -2\,\gamma^\nu$
- $\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$

(v) Is there an easier way to prove finiteness of light-by-light scattering assuming gauge invariance? If yes, show it.

3. The renormalised photon propagator

The goal of this exercise is to understand the renormalised photon propagator (14.33),

$$
G_{\mu\nu}(k) \;=\; \langle \Omega, \mathrm{T}[\mathcal{A}_\mu(x)\mathcal{A}_\nu(y)]\,\Omega\rangle .
$$

The vacuum polarisation tensor is given by

$$
\mathrm{i}\Pi^{\mu\nu}(k) \;=\; \omega_R(k^2,\Lambda)\left[g^{\mu\nu}k^2 - k^\mu k^\nu\right],
$$

(14.31).

Using the diagrammatic expansion (14.34), show that

$$
G_{\mu\nu}(k) \;=\; \frac{-\mathrm{i}}{k^2\left(1+\omega_R(k^2,\Lambda)\right) - \mu^2 + \mathrm{i}0}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right).
$$

Why can one neglect the term proportional to $k_\mu k_\nu$?

Show that, for small momenta k, $G_{\mu\nu}(k)$ is given by the free photon propagator

$$
-\mathrm{i}\widehat{D}_{\mu\nu}(k) \;=\; \frac{\mathrm{i}g_{\mu\nu}}{k^2 - \mu^2 + \mathrm{i}0} .
$$