Quantum Field Theory I, Exercise Set 8.

HS 08 *Due:* 20/21 November 2008

1. The Green's function in the interaction picture

We want to verify equation (12.31) in the lecture notes:

$$\langle \Omega \,|\, \mathrm{T}\left[\varphi(x_1)\cdots\varphi(x_n)\right] \,|\, \Omega \rangle = \lim_{t\to\infty} Z^{-1}_{[-t,t]} \langle 0 \,|\, \mathrm{T}\left[\varphi_0(x_1)\cdots\varphi_0(x_n) e^{-\mathrm{i}\int_{-t}^{t} \mathrm{d}\tau\, H_W(\tau)}\right] \,|\, 0 \rangle \ . \qquad (1)$$

The $\varphi(x)$ are the field operators in the Heisenberg picture while $\varphi_0(x)$ are free field operators.

We define $H = H_0 + H_W$, where H_0 describes the free dynamics, and assume that we have normalised the energy so that $H|\Omega\rangle = 0$. The propagator in the interaction picture is $U_W(t,s) = \mathrm{T}\left[\exp -i\int_s^t \mathrm{d}\tau\, H_W(\tau)\right]$ and the following relations hold:

$$\begin{aligned} e^{-\mathrm{i}(t-s)H} &= e^{-\mathrm{i}tH_0} U_W(t,s) e^{\mathrm{i}sH_0} \ , \\ e^{\pm\mathrm{i}tH_0}|0\rangle &= |0\rangle \ , \\ e^{\mathrm{i}x^0 H_0}\varphi(\vec{x})e^{-\mathrm{i}x^0 H_0} &= \varphi_0(x^0,\vec{x}) = \varphi_0(x) \ . \end{aligned}$$

In the following, if not mentioned otherwise, we assume the spacetime points x_i, $1 \le i \le n$, have time components t_i ordered so that $t > t_1 > t_2 > \ldots > t_n > -t$.

(i) Show that the following identity for time ordered products holds:

$$\mathrm{T}\left[e^A e^B\right] = \mathrm{T}e^{A+B} \ .$$

Hint: Write the exponential as a series and exploit the time order operator to reshuffle the series.

(ii) Show that

$$\varphi(x_1)\cdots\varphi(x_n) = U_W(0,t_1)\varphi_0(x_1)U_W(t_1,t_2)\varphi_0(x_2)\cdots\varphi_0(x_n)U_W(t_n,0) \ .$$

Hint: Remember the time evolution of operators in the Heisenberg picture.

(iii) Use $|\Omega\rangle = \lim_{t\to\pm\infty} c_t e^{\mathrm{i}tH}|0\rangle$ and $U_W(t,\tau)U_W(\tau,s) = U_W(t,s)$ to show that

$$\begin{aligned} &\langle \Omega \,|\, \mathrm{T}\left[\varphi(x_1)\cdots\varphi(x_n)\right] \,|\, \Omega \rangle \\ &\qquad = \lim_{t\to\infty} |c_t|^2 \langle 0 \,|\, \mathrm{T}\left[e^{-\mathrm{i}tH_0}U_W(t,t_1)\varphi_0(x_1)\cdots\varphi_0(x_n)U_W(t_n,-t)e^{-\mathrm{i}tH_0}\right] \,|\, 0 \rangle \ . \end{aligned}$$

(iv) Show that

$$\langle \Omega \,|\, A \,|\, \Omega \rangle = \lim_{t\to\infty}\lim_{s\to-\infty} \frac{\langle 0 \,|\, e^{\mathrm{i}H_0 t}e^{-\mathrm{i}Ht}Ae^{\mathrm{i}Hs}e^{-\mathrm{i}H_0 s} \,|\, 0 \rangle}{\langle 0 \,|\, U_W(t,s) \,|\, 0 \rangle} \ .$$

Hint: Start with $|\Omega\rangle = \lim_{t\to\pm\infty} \frac{e^{\mathrm{i}Ht}|0\rangle}{\langle \Omega | 0 \rangle}$, then determine $|\langle \Omega \,|\, 0 \rangle|^2$ by setting $A = \mathbb{1}$.

(v) Identify $\lim_{t\to\infty} |c_t|^2$ with $Z_{[-t,t]}^{-1}$ and conclude from (iv) that

$$Z_{[-t,t]} = \langle 0 \,|\, U_W(t,-t) \,|\, 0 \rangle \ .$$

Express $U_W(t,s)$ by its representation as time ordered exponential and show (1).

2. Wick's theorem

In class, Wick's theorem was introduced as a tool for expressing products of field operators as sums of Wick-ordered expressions. It was proven by induction. Here we discuss a different derivation.

We consider a free field theory, described by the field operators $\varphi_0(x)$, or, equivalently, by the free creation and annihilation operators. The free vacuum is denoted by $|0\rangle$. For simplicity, we consider bosonic fields (fermionic fields may be dealt with similarly with the additional complication of having to keep track of signs).

Let $A_1, \ldots, A_n$ be linear combinations of creation and annihilation operators. For example, A_i could be of the form $\varphi_0(x_i)$. The *Wick-ordered product* $: A_1 \cdots A_n :$ is defined by multiplying out $A_1 \cdots A_n$ and writing the creation operators on the left of the annihilation operators. Wick-ordered expressions have the important property

$$\langle 0 \,|\, : A_1 \cdots A_n : \,|\, 0 \rangle \ = \ 0 \,.$$

(i) Let $0 \le p \le \lfloor n/2 \rfloor$ and denote by $P : (i_1 < j_1) \cdots (i_p < j_p)$ a *pairing*, i.e. a choice of p disjoint pairs from the set $\{1, \ldots, n\}$. Show that

$$A_1 \cdots A_n \ = \ \sum_{P:(i_1<j_1)\cdots(i_p<j_p)} \left[\prod_{l=1}^{p} \langle 0 \,|\, A_{i_l} A_{j_l} \,|\, 0 \rangle \right] \, : A_1 \cdots \widehat{A}_{i_1} \cdots \widehat{A}_{j_p} \cdots A_n :$$

where $\widehat{\cdot}$ denotes omission.
Hint: Since both sides are multilinear in the variables $A_1, \ldots, A_n$, one can assume that each A_i is either a creation or annihilation operator.

(ii) One often needs to compute the Wick-ordered product of the product of Wick-ordered products. Show that

$$: A_1 \cdots A_k : \, : A_{k+1} \cdots A_n : \ = \ \sum_{P:(i_1<j_1)\cdots(i_p<j_p)}{}' \left[\prod_{l=1}^{p} \langle 0 \,|\, A_{i_l} A_{j_l} \,|\, 0 \rangle \right] \, : A_1 \cdots \widehat{A}_{i_1} \cdots \widehat{A}_{j_p} \cdots A_n :$$

where the primed sum means that one only sums over pairings where no pair $\{i_l, j_l\}$ originates from the same Wick-ordered product.

What is the generalisation to an arbitrary number of terms?

(iii) In perturbation theory one computes time-ordered products. Let T denote time-ordering. Generally, T may be expressed as

$$\mathrm{T}(A_1 \cdots A_n) \ = \ A_{\sigma(1)} \cdots A_{\sigma(n)} \,,$$

where σ is some permutation (the explicit expression for σ is not needed). Show that

$$: \mathrm{T}(A_1 \cdots A_n) : \, = \, : A_1 \cdots A_n : \ .$$

(iv) Show that

$$\mathrm{T}(A_1 \cdots A_n) = \sum_{P:(i_1<j_1)\cdots(i_p<j_p)} \left[\prod_{l=1}^{p} \langle 0 \,|\, \mathrm{T}(A_{i_l} A_{j_l}) \,|\, 0\rangle\right] : A_1 \cdots \widehat{A}_{i_1} \cdots \widehat{A}_{j_p} \cdots A_n : \,.$$

Hints: Use (i) and write $A_{\sigma(i)} = B_i$.

(v) Show that

$$\mathrm{T}(: A_1 \cdots A_k :: A_{k+1} \cdots A_n :) = {\sum_{P:(i_1<j_1)\cdots(i_p<j_p)}}' \left[\prod_{l=1}^{p} \langle 0 \,|\, \mathrm{T}(A_{i_l} A_{j_l}) \,|\, 0\rangle\right] : A_1 \cdots \widehat{A}_{i_1} \cdots \widehat{A}_{j_p} \cdots A_n : \,.$$

Hints: Use (ii) and proceed like in (iv).

What is the generalisation to an arbitrary number of terms?

3. Feynman diagrams for φ^4

Consider the φ^4–QFT with Lagrangian density

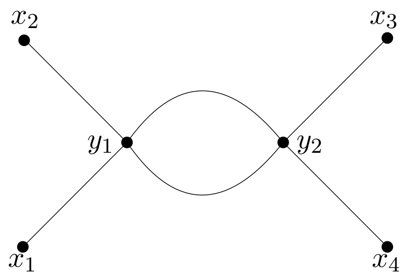
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 \,.$$

We consider two particle scattering of the form $b + b \to b + b$.

(i) Compute the differential cross section in the CoM frame for elastic scattering to lowest order in perturbation theory.

Hint: Replace $Z = 1 + O(\lambda^2)$ by 1 in the LSZ reduction formula.

(ii) Show that the Feynman diagram below is logarithmically divergent.



(iii) Find the symmetry factor for the following Feynman diagram.