Quantum Field Theory I, Exercise Set 7.

HS 08 *Due:* 13/14 November 2008

1. Scattering matrix

In the lecture notes the asymptotic Hilbert spaces $\mathcal{H}_{\rm as}$, where 'as' stands for 'in' or 'out', were constructed by applying the asymptotic creation operators $\alpha^*_{\rm as}(\mathbf{p})$ to the vacuum state $|0\rangle_{\rm in} = \Omega = |0\rangle_{\rm out}$. The asymptotic completeness assumption states that

$$\mathcal{H}_{\rm out} = \mathcal{H}_{\rm in} = \mathcal{H}_{\rm int}\,,$$

where $\mathcal{H}_{\rm int}$ is the Hilbert space of the interacting theory. The scattering amplitude is then of the form

$$\left\langle \alpha^*_{\rm out}(\mathbf{p}_1)\ldots\alpha^*_{\rm out}(\mathbf{p}_n)\Omega\,,\alpha^*_{\rm in}(\mathbf{k}_1)\ldots\alpha^*_{\rm in}(\mathbf{k}_m)\Omega\right\rangle\,,$$

which can be computed using the LSZ reduction formula. The scattering matrix is defined as the map

$$\begin{gathered} S^*\,:\,\mathcal{H}_{\rm in}\to\mathcal{H}_{\rm out} \\ \alpha^*_{\rm in}(\mathbf{p}_1)\ldots\alpha^*_{\rm in}(\mathbf{p}_n)\Omega\mapsto\alpha^*_{\rm out}(\mathbf{p}_1)\ldots\alpha^*_{\rm out}(\mathbf{p}_n)\Omega \end{gathered}$$

and $S^*\Omega = \Omega$.

(i) Assuming asymptotic completeness, prove that S is unitary.

(ii) Show that
$$S^*\alpha^{\#}_{\rm in}(\mathbf{p})S = a^{\#}_{\rm out}(\mathbf{p})\,.$$

2. Asymptotic states and scattering of two particles

Consider a field theory with particle mass m.

(i) We take two incoming particles with momenta $\mathbf{k}_1, \mathbf{k}_2$ and look at scattering amplitudes corresponding to more than 2 outgoing particles, i.e.

$$\left\langle \alpha^*_{\rm out}(\mathbf{p}_1)\cdots\alpha^*_{\rm out}(\mathbf{p}_n)\Omega\,,\alpha^*_{\rm in}(\mathbf{k}_1)\alpha^*_{\rm in}(\mathbf{k}_2)\Omega\right\rangle\,, \tag{1}$$

where $n \geq 3$. Show that the 'disconnected terms' in the LSZ reduction formula vanish.
Hint: By (10.19), one has to show that

$$\left\langle \alpha^*_{\rm out}(\mathbf{p}_2)\cdots\alpha^*_{\rm out}(\mathbf{p}_n)\Omega\,,\alpha^*_{\rm in}(\mathbf{k}_2)\Omega\right\rangle$$

vanishes. To this end, show that

$$\sum_{i=2}^n \omega(\mathbf{p}_i) \;>\; \omega\Big(\sum_{i=2}^n \mathbf{p}_i\Big)\,,$$

where $\omega(\mathbf{p}) \;=\; \sqrt{m^2+\mathbf{p}^2}$.

(ii) Show that

$$\alpha^*_{\text{in}}(\mathbf{p})\Omega = \alpha^*_{\text{out}}(\mathbf{p})\Omega\,.$$

Hint: Consider projections onto states of the form $\alpha^*_{\text{in}}(\mathbf{p}_1)\cdots\alpha^*_{\text{in}}(\mathbf{p}_n)\Omega$. Treat the cases $n=0$, $n=1$, $n>1$ separately.

3. Phase space factor

In this exercise we consider $2\to 2$ scattering. Starting from the expression for the differential cross section

$$\mathrm{d}\sigma = (2\pi)4\delta^{(4)}(p_1+p_2-p_1'-p_2')\frac{1}{4\big((p_1\cdot p_2)^2-m_1^2m_2^2\big)^{\frac{1}{2}}}\left|\mathcal{M}(p_1,p_2,p_1',p_2')\right|^2\frac{\mathrm{d}\mathbf{p}_1'}{(2\pi)^3\omega_1'}\frac{\mathrm{d}\mathbf{p}_2'}{(2\pi)^3\omega_2'}$$

show that in a frame where $\mathbf{p}_1$ and $\mathbf{p}_2$ are parallel, e.g. in the center of mass frame,

$$\mathrm{d}\sigma = M(p_1,p_2,p_1',p_2')\,\delta^{(4)}(p_1+p_2-p_1'-p_2')\,\mathrm{d}\mathbf{p}_1'\,\mathrm{d}\mathbf{p}_2'$$

with invariant transition amplitude

$$M(p_1,p_2,p_1',p_2') = \frac{1}{16\pi^2}\frac{1}{v_{\text{rel}}\,\omega_1\,\omega_2\,\omega_1'\,\omega_2'}\left|\mathcal{M}\right|^2\,.$$

Here we have used the notation $v_{\text{rel}} = \left|\frac{\mathbf{p}_1}{\omega_1}-\frac{\mathbf{p}_2}{\omega_2}\right|$ with $\omega_i=\omega(\mathbf{p}_i)$.