Quantum Field Theory I, Exercise Set 2

HS 08 *Due:* 9/10 October 2008

1. Contour integral representation of propagators

Prove equations (3.35), (3.36) and (3.37) in the lecture notes.

2. Properties of the free scalar field

Consider a scalar field given by

$$\varphi(x) \;=\; \int \mathrm{d}^3p \,\Big\{ f(\vec{p})\, a^*(\vec{p})\, \mathrm{e}^{\mathrm{i}(\omega(\vec{p})t - \vec{p}\cdot\vec{x})} + \text{h.c.} \Big\},$$

where $a^*(\vec{p})$ and $a(\vec{p})$ satisfy the canonical commutation relations, and f and ω are some functions on $\mathbb{R}^3$.

(i) We require the field to describe a free particle of mass m, i.e. φ should satisfy the Klein-Gordon equation

$$(\Box + m^2)\varphi(x) \;=\; 0\,.$$

What does this imply for ω?

(ii) We additionally require the field to satisfy Poincaré covariance:

$$U(\Lambda, a)\,\varphi(x)\,U(\Lambda, a)^{-1} \;=\; \varphi(\Lambda(x+a))\,. \tag{1}$$

Show that this implies that

$$\mathrm{i}\Delta(x-y) \;=\; [\varphi(x), \varphi(y)]$$

is Lorentz invariant.

(iii) Show that the Lorentz invariance of Δ implies that, up to a phase, f is equal to $\frac{c}{\sqrt{2\omega(\vec{p})}}$, where $c > 0$ is some constant.

(iv) Show that $c = 1$ if we impose the canonical commutation relations (3.31)

$$[\pi(0,\vec{x}), \varphi(0,\vec{y})] \;=\; -\mathrm{i}\delta(\vec{x}-\vec{y})\,.$$

(v) In class it was shown that the propagator Δ satisfies causality, i.e. vanishes for space-like arguments. Replace now the bosonic creation and annihilation operators $a(\vec{p})^*, a(\vec{p})$ by fermionic creation and annihilation operators, which satisfy the canonical anticommutation relations

$$\{a(\vec{p}), a(\vec{p}\,')\} \;=\; \{a^*(\vec{p}), a^*(\vec{p}\,')\} \;=\; 0\,, \qquad \{a(\vec{p}), a^*(\vec{p}\,')\} \;=\; \delta(\vec{p}-\vec{p}\,')\,.$$

Show that causality cannot hold, i.e. the anticommutator $\{\varphi(x), \varphi(y)\}$ does not vanish for space-like arguments. Thus, the free scalar field describes bosons. This is a special case of the celebrated *spin-statistics theorem.*

(vi) Using the Poincaré invariance (1) of the scalar field $\varphi(x)$ and

$$\varphi(x) \;=\; \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3p}{\sqrt{2\omega(\vec{p})}} \Big\{ a^*(\vec{p})\, \mathrm{e}^{\mathrm{i}(\omega(\vec{p})x^0 - \vec{p}\cdot\vec{x})} + a(\vec{p})\, \mathrm{e}^{-\mathrm{i}(\omega(\vec{p})x^0 - \vec{p}\cdot\vec{x})} \Big\}$$

(equation (3.21) in the lecture notes), derive the explicit expression for the action of the Poincaré group on Fock space:

$$U(\Lambda, a)\, a^*(\vec{p})\, U(\Lambda, a)^{-1} \;=\; \mathrm{e}^{\mathrm{i}(\omega(\vec{p})a^0 - \vec{p}\cdot\vec{a})}\, a^*(\overrightarrow{\Lambda p}) \sqrt{\frac{\omega(\overrightarrow{\Lambda p})}{\omega(\vec{p})}}\,,$$

(equation (3.20) in the lecture notes).

3. Representations of the Dirac algebra

Consider the Dirac algebra generated by the elements $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ satisfying the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} \;=\; 2g_{\mu\nu}\,.$$

Here g is the Minkowski metric $g = \mathrm{diag}(1, -1, -1, -1)$.

(i) We choose a new basis of the Dirac algebra, $\{a_0, a_1, a_0^*, a_1^*\}$, defined through

$$\gamma_0 \;=\; a_0^* + a_0\,, \qquad \gamma_1 \;=\; \mathrm{i}(a_1^* + a_1)\,, \qquad \gamma_2 \;=\; a_0 - a_0^*\,, \qquad \gamma_3 \;=\; a_1 - a_1^*\,.$$

Show that the elements a_0, a_1, a_0^*, a_1^* satisfy the canonical anticommutation relations

$$\{a_\mu, a_\nu\} \;=\; \{a_\mu^*, a_\nu^*\} \;=\; 0\,, \qquad \{a_\mu, a_\nu^*\} \;=\; \delta_{\mu\nu}\,, \qquad (\mu = 0, 1).$$

(ii) Define the "particle number operator" $n_\mu = a_\mu^* a_\mu$ for $\mu = 0, 1$. Show that n_0 and n_1 commute, and that

$$n_\mu \;=\; n_\mu^2\,.$$

(iii) Show that the only irreducible representation of the Dirac algebra is four-dimensional.
Hint: Diagonalise n_0 and n_1, and pick a vector $|0, 0\rangle$ with eigenvalues 0. Study the action of the Dirac algebra on $|0, 0\rangle$.

4. Discrete symmetries of the Dirac equation

Consider the Dirac equation in the presence of an external electromagnetic field A_μ:

$$\big(\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu) - m\big)\psi \;=\; 0\,.$$

We work in the "chiral representation"

$$\gamma^0 \;=\; \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \qquad \gamma^i \;=\; \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix},$$

where $\sigma_0 = \mathbb{1}$ and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. The goal of this exercise is to determine explicit expressions for the three discrete symmetries $\mathbb{P}, \mathbb{T}, \mathbb{C}$ of the Dirac equation.

(i) The parity transformation $\mathbb{P}$ is of the form

$$(\mathbb{P}\psi)(x) \;=\; U_{\mathbb{P}}\psi(Px)\,,$$

where $P(t, \vec{x}) \;=\; (t, -\vec{x})$, and $U_{\mathbb{P}}$ is an operator on $\mathbb{C}^4$. Determine $\mathbb{P}$ from the requirement that

$$\big(\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu) - m\big)\psi \;=\; 0 \qquad \Longleftrightarrow \qquad \big(\gamma^\mu(\mathrm{i}\partial_\mu - e\tilde{A}_\mu) - m\big)\mathbb{P}\psi \;=\; 0\,,$$

where $\tilde{A}(x) = PA(Px)$.

(ii) The time-reversal transformation $\mathbb{T}$ is of the form

$$(\mathbb{T}\psi)(x) \;=\; U_{\mathbb{T}}\psi(Tx)\,,$$

where $T(t, \vec{x}) \;=\; (-t, x)$. Determine $\mathbb{T}$ from the requirement that

$$\big(\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu) - m\big)\psi \;=\; 0 \qquad \Longleftrightarrow \qquad \big(\gamma^\mu(\mathrm{i}\partial_\mu - e\hat{A}_\mu) - m\big)\mathbb{T}\psi \;=\; 0\,,$$

where $\hat{A}(x) = PA(Tx)$.

(iii) The charge conjugation $\mathbb{C}$ is of the form

$$(\mathbb{C}\psi)(x) \;=\; U_{\mathbb{C}}\psi(x)\,.$$

Determine $\mathbb{C}$ from the requirement that

$$\big(\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu) - m\big)\psi \;=\; 0 \qquad \Longleftrightarrow \qquad \big(\gamma^\mu(\mathrm{i}\partial_\mu + eA_\mu) - m\big)\mathbb{C}\psi \;=\; 0\,.$$