General relativity, exercise sheet 9.

HS 08 *Due:* Fri, November 28, 2008

1. Radiation dominated universe (RD)

The model of a homogeneous and isotropic universe (Friedmann model) was treated in class in the case where $T^{\mu\nu}$ is the energy-momentum tensor of dust (matter dominated universe (MD)). Consider now a homogeneous isotropic radiation field

$$T^{\mu\nu} = \frac{\rho}{3}\,(4\,u^\mu u^\nu - g^{\mu\nu}) \qquad (c=1)\,,$$

with $u^\mu = \delta^\mu{}_\nu$ at $(t,0,0,0)$ w.r.t. the usual chart.

i) Consider the Friedmann equations (6.13, 6.14) in this case. Show that $\rho a^4/3$ is conserved.

ii) Find the solutions $a(t)$ for $\Lambda = 0$.

Hint: Introduce instead of t the new coordinate

$$\eta = \int_0^t \frac{dt'}{a(t')}\,. \tag{1}$$

2. The causal structure of the Friedmann models

Using the above conformal time η, the metric of the Friedmann models reads (see 6.5)

$$g = a^2(\eta)[d\eta^2 - (d\chi^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2))]\,,$$

where $r = r(\chi)$ was defined in (6.4).

i) Compute the range $\eta \in [0,\eta_0]$ in the cases MD and RD.

ii) Let $k = 1$. Consider null geodesics ending at $\chi = 0$ in an $(\eta,\chi) \in [0,\eta_0]\times[0,\pi]$ diagram. Is it possible for an observer to send light signals that come back to him?