General relativity, exercise sheet 4.

HS 08 *Due:* Fri, October 24, 2008

1. Euclidean metric in polar coordinates

Consider the Euclidean plane as a Riemannian manifold $M = \mathbb{R}^2 \ni (x^1, x^2) = x$ with metric $g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$. Compute the metric in polar coordinates r, φ and the Christoffel symbols (3.6) of the Riemann connection. Verify that they agree with those computed in Exercise 3.1.

2. Geodesics in the hyperbolic plane

Consider the hyperbolic plane: $\mathbb{H}^2 = \{(x, y) | y > 0\}$ with the metric $g = y^{-2}(dx \otimes dx + dy \otimes dy)$.

a) Write the geodesic equation.

b) Find the quantities which are conserved along the geodesics.

c) Show that the geodesics are the Euclidean half-circles, centered on the line $y = 0$.

Hint: Show that the curvature

$$\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$

is constant.

3. An affine connection on Lie groups

Consider a Lie Group (see Exercise 2.2).

a) Show that there is a unique affine connection ∇ on G with the properties that

i) for any left-invariant vector field V on G, the tangent vectors $d\gamma/dt = V_{\gamma(t)}$ to any of its orbits $\gamma(t)$ are parallel transported along it;

ii) the torsion vanishes.

To define a connection ∇ is tantamount to prescribing its coefficients $\langle e^\alpha, \nabla_{e_\beta} e_\gamma \rangle$, see (4.12), w.r.t. vector fields e_α, resp. 1-forms e^β forming dual bases $(e_1, \dots e_n)$, $(e^1, \dots e^n)$, but not necessarily coordinate bases.

b) Show that the connection of part (a) has coefficients

$$\langle e^\gamma, \nabla_{e_\alpha} e_\beta \rangle = \frac{1}{2} C^\gamma{}_{\alpha\beta}\,, \tag{1}$$

where the e_α are left-invariant basis fields and $C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$ their structure constants (see Exercise 2.2 iv)).

Hints:

- What are the equations for ∇ expressing (i,ii)? Rewrite them in terms of a left-invariant basis.
- Consider the parts of (1) which are symmetric, resp. antisymmetric in α, β.
- Prove (a,b) at once by showing that the equations in the two first hints are equivalent.