The homogeneous, isotropic universe

Time slice : 3-dim. mfld $M_0 \subset \mathbb{R}^4 \ni (x^1, .. x^4)$
of constant curvature

$$M_0 : \; k[(x^1)^2 + (x^2)^2 + (x^3)^2] + (x^4)^2 = R_0^2$$

$$g_0 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + k\,(dx^4)^2$$

with $R_0 > 0$ and

$$k_0 = \begin{cases} +1 & \text{3-sphere} \\ 0 & \text{Euclidean 3-space} \\ -1 & \text{3-hyperboloid} \end{cases}$$

Charts:

A : coordinates (x^1, x^2, x^3)
coordinate map $x^4 = \sqrt{R_0^2 - kr^2} \equiv w(r)$
with $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$

$$g_0 = \sum_{i=1}^{3} (dx^i)^2 + \frac{k}{R_0^2 - kr^2} \sum_{i,j=1}^{3} x^i x^j \, dx^i dx^j$$

B : coord (r, θ, φ) : spherical coordinates for (x^1, x^2, x^3)

$$g_0 = R_0^2 \frac{(dr)^2}{w^2} + r^2 \left((d\theta)^2 + \sin^2\theta \, (d\varphi)^2\right)$$

Consequences

- Two such observers have distance at time

$$d(t) = a(t)\, d_0$$

(d_0 : distance w.r.t. g_0)

$$\rightarrow \quad \frac{\dot d(t)}{d(t)} = \frac{\dot a(t)}{a(t)} =: H(t)$$

is independent of pair of observers:

$$\dot d(t) = H(t)\, d(t) \qquad \text{(Hubble law)}$$

$H(t)$: Hubble "constant" (expansion rate)

Today:

$$H(t_0) = 71 \pm 4 \ \frac{km/s}{Mpc}$$

Setting up the field equations

- matter: ideal fluid

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}$$

where $p = p(\rho)$ (eq. of state)

$\rho = \rho(t)$

$u^\mu = (1, 0, 0, 0)$

- geometry

– redundancy: $(R_0, a(t)) \sim (\lambda R_0, \lambda^{-1} a(t))$

set $R_0 = 1$ (replace $k/R_0^2 \rightsquigarrow k$)

– Christoffel symbols (others = 0)

$$\Gamma^0_{ii} = a\dot{a}$$

$$\Gamma^i_{i0} = \Gamma^i_{0i} = \frac{\dot{a}}{a}$$

$$\Gamma^i_{ee} = kx^i$$

– Ricci tensor

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$$

$$= \Gamma^\alpha{}_{\nu\mu,\alpha} - \Gamma^\alpha{}_{\alpha\mu,\nu} + \Gamma^\sigma{}_{\nu\mu}\Gamma^\alpha{}_{\alpha\sigma} - \Gamma^\sigma{}_{\alpha\mu}\Gamma^\alpha{}_{\nu\sigma}$$

Killing fields

(M, g) pseudo-Riemannian manifold

$\varphi_s : M \to M$ flow of isometries:

$$\varphi_s^* g = g$$

Let K be the generating vector field of φ_s. Then

$$\mathcal{L}_K g = \frac{d}{ds}(\varphi_s^* g)\Big|_{s=0} = 0$$

Def. A vectorfield K with

$$\mathcal{L}_K g = 0$$

is a Killing field for g.

Christoffel symbols : $\neq 0$ are only $\left(' = \frac{d}{dr} \right)$

$$\Gamma^t_{tr} = \Gamma^t_{rt} = a'$$

$$\Gamma^r_{tt} = a' e^{2(a-b)} \quad , \quad \Gamma^r_{rr} = b'$$

$$\Gamma^r_{\theta\theta} = -r e^{2b} \quad , \quad \Gamma^r_{\varphi\varphi} = r \sin^2\theta \, e^{-2b}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = r^{-1} , \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta \cos\theta$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = r^{-1} , \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta$$

Ricci tensor: diagonal with

$$R_{tt} = -(a'b' - a'' - a'^2) e^{2(a-b)} + \frac{2a'}{r} e^{2(a-b)}$$

$$R_{rr} = (a'b' - a'' - a'^2) + \frac{2b'}{r}$$

$$R_{\theta\theta} = r(b' - a') e^{-2b} + 1 - e^{-2b}$$

$$R_{\varphi\varphi} = (\sin^2\theta) R_{\theta\theta}$$

The Schwarzschild metric

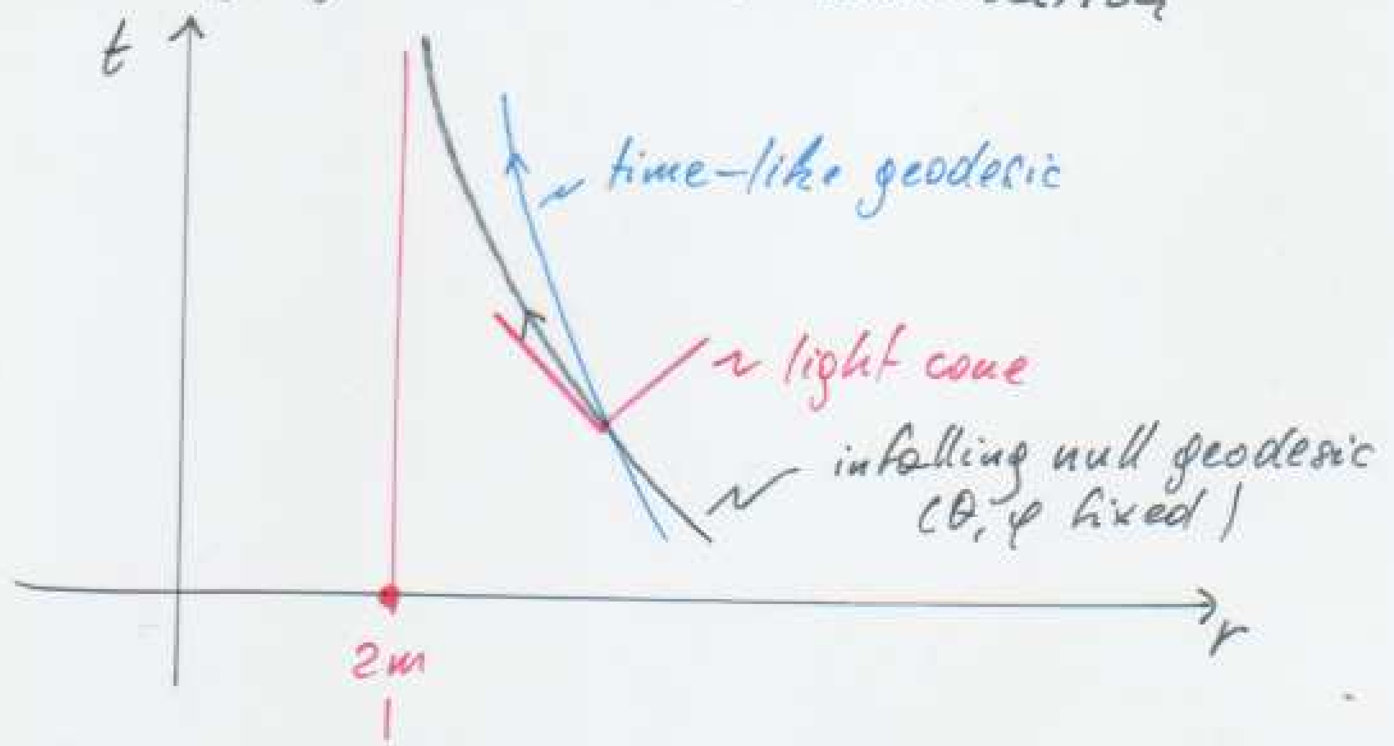
$$g = \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

domain of chart:

$$t \in \mathbb{R}, \quad r > 2m, \quad \vec{e} \in S^2, \quad \vec{e} = \vec{e}(\theta, \varphi)$$

$\uparrow$ Schwarzschild radius

Describes (static) metric in the exterior of a spherically symmetric mass distribution



Null-directions: $\quad 0 = \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2$

$$\frac{dt}{dr} = \pm\left(1-\frac{2m}{r}\right)^{-1}$$

Infalling null geodesic: $\quad dt = -\frac{r}{r-2m}\,dr \rightarrow \int_{r_0}^{r \downarrow 2m} dt = \infty$

even more so for time-like geodesics

Geodesics in the Schwarzschild metric

radial eq. ($\cdot = d/d\tau$; τ affine parameter)

$$\dot{r}^2 + \left(1 - \frac{2m}{r}\right)\left(\mathcal{L} + \frac{\ell^2}{r^2}\right) = \mathcal{E}^2$$

product $\mathcal{L}$: GR correction

- time-like geodesics

$\mathcal{L} = 1 \iff \tau$: proper time

- null geodesics

$\mathcal{L} = 0$

---

trajectories: find $r = r(\varphi)$, resp. $u = u(\varphi)$
with $u = 1/r$

$$u'' + u - \mathcal{L}\frac{m}{\ell^2} = 3mu^2$$

Perihelion advance

$(\mathcal{L}=1)$

$$u'' + u - \frac{m}{\ell^2} = \underbrace{3mu^2}_{\text{GR correction}} \qquad (*)$$

$\Big\lceil$ $u''+u-\frac{m}{\ell^2}=0$ has solution

$$u_0 = \frac{1}{d}(1+\varepsilon\cos\varphi) \qquad d = \ell^2/m, \quad \varepsilon > 0$$

for normalization: perihelion at $\varphi = 0$ $\Big\rfloor$

Perturbative ansatz for $(*)$: $u = u_0 + v$

$$\rightsquigarrow \quad v'' + v = 3mu_0^2 = \frac{3m}{d^2}(1+2\varepsilon\cos\varphi + \varepsilon^2\cos^2\varphi)$$

with $v = v' = 0$ at $\varphi = 0$ (perihelion fixed)

$$v''+v = \begin{cases} A_1 \\ A_2\cos\varphi \\ A_3\cos^2\varphi \end{cases} \longrightarrow v = \begin{cases} A_1(1-\cos\varphi) \\ \frac{1}{2}A_2\,\varphi\sin\varphi \\ A_3\left(\frac{1}{2}+\frac{1}{6}\cos 2\varphi - \frac{2}{3}\cos\varphi\right) \end{cases}$$

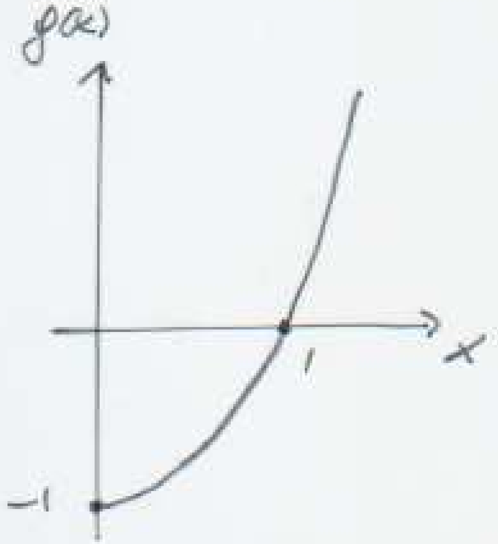
$$ds^2 = \frac{32m^3}{r} e^{-\frac{r}{2m}} (dv^2 - du^2)$$

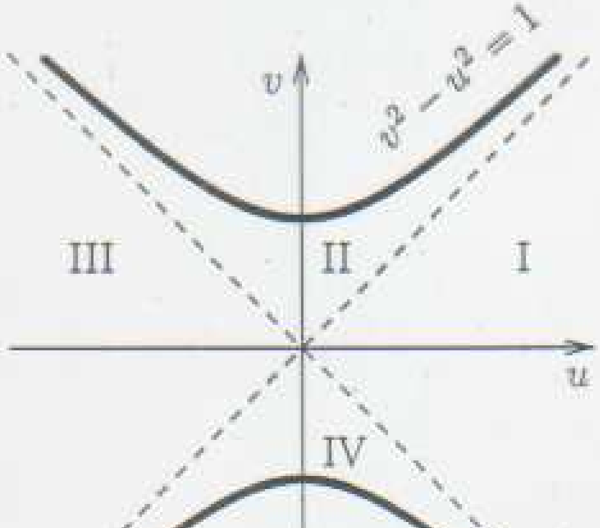
$$+ \text{ angular part}$$

where $r = r(u, v)$ is the solution of

$$g\left(\frac{r}{2m}\right) = u^2 - v^2$$

$$g(x) = (x-1)\, e^{-x}$$





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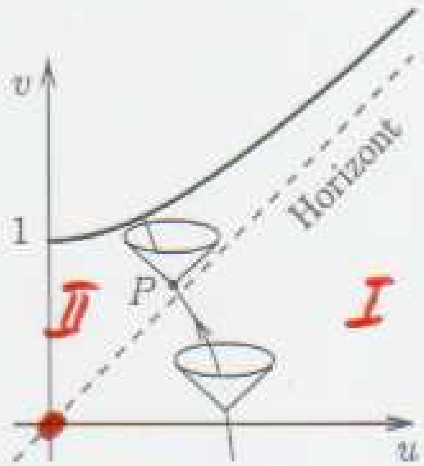
v² − u² = 1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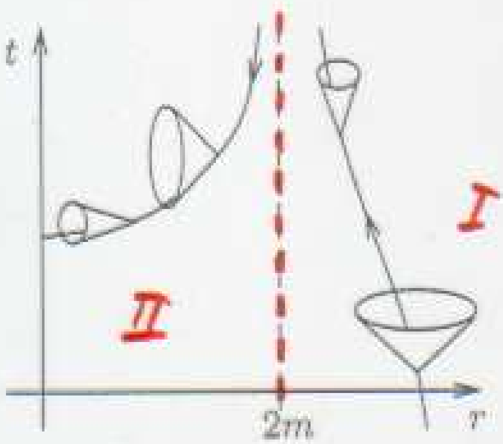
t
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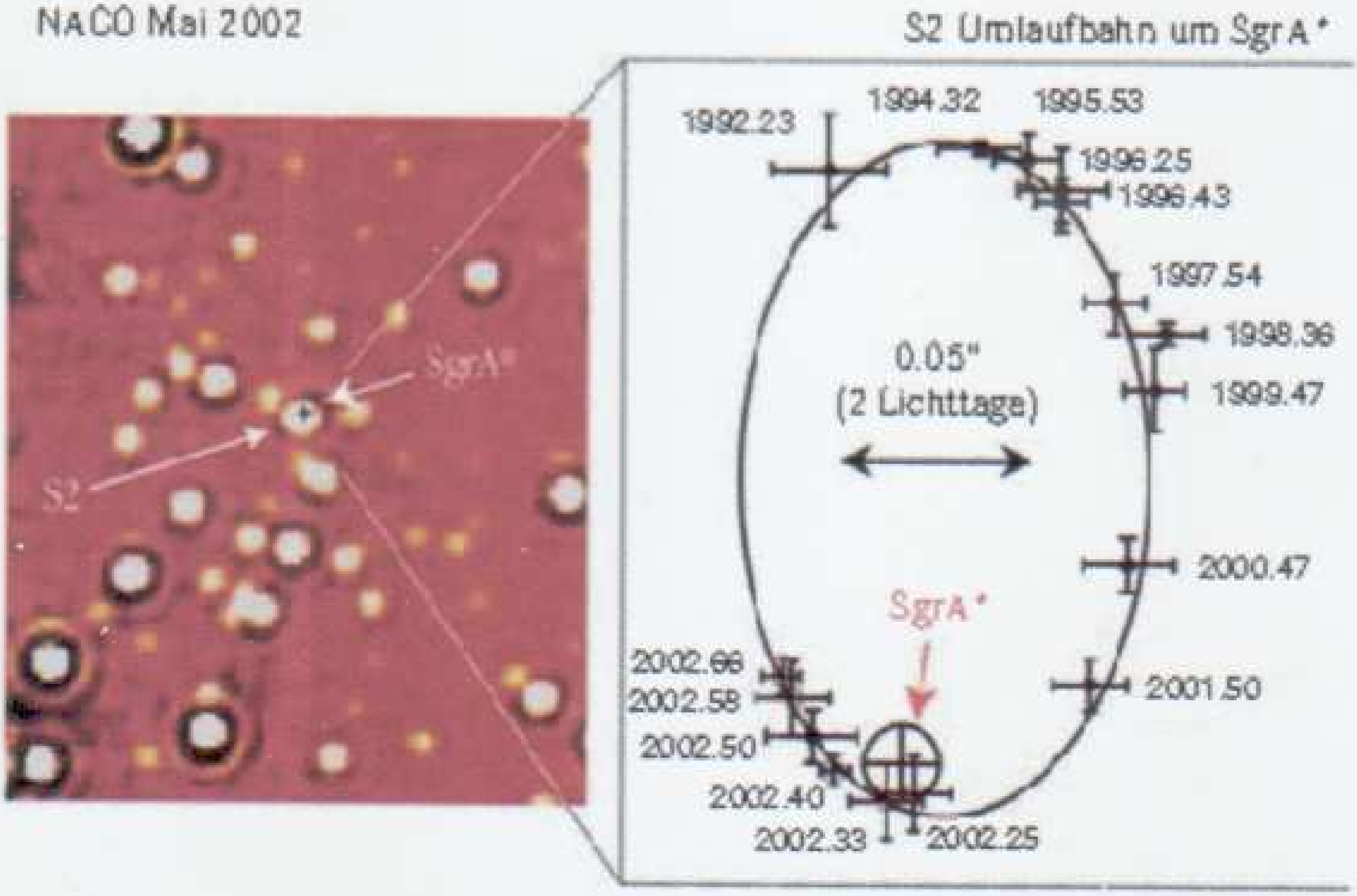
Motion of a star (S2) around the central black hole in the Milky Way



$$T = \frac{2\pi}{\sqrt{G_0 M}}\, a^{3/2} \xrightarrow[\text{known}]{a,\,T} M \stackrel{!}{=} 2.6 \cdot 10^6 M_\odot$$

The Kerr metric

- domain of chart (t, r, θ, φ):

  $t \in \mathbb{R}$; $r > r_+$; θ, φ spherical coordinates
- parameters: m, a
- notations

$$\Delta = r^2 - 2mr + a^2$$

$$\rho^2 = r^2 + a^2 \cos^2\theta$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$$

- metric

$$ds^2 = \left(1 - \frac{2m}{\rho^2}\right) dt^2 + \frac{4mar}{\rho^2} \sin^2\theta \, d\varphi \, dt$$

$$- \frac{\Sigma^2}{\rho^2} \sin^2\theta \, d\varphi^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

$$= \frac{\rho^2}{\Sigma^2} \Delta \, dt^2 - \frac{\Sigma^2}{\rho^2} \sin^2\theta \, (d\varphi - \Omega \, dt)^2$$

$$- \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

with

$$\Omega = a \cdot \frac{2mr}{\Sigma^2}$$

- stationary, axisymmetric solution of vacuum equation.

- meaning of parameters

  m: mass of black hole

  $J = a \cdot m$: angular momentum

- metric components have singularity at

$$\Delta = 0$$

i.e. $r = r_\pm = m \pm \sqrt{m^2 - a^2}$. Hence forth:

$$r > r_+$$

- Killing fields: $\Phi = \frac{\partial}{\partial \varphi}$, $K = \frac{\partial}{\partial t}$

  * Φ is spacelike

  * K is timelike for

$$r > r_0(\theta) := m + \sqrt{m^2 - a^2 \cos^2\theta}$$

$$(\geq r_+)$$

On the ergosphere

Lewis Carroll, Through the Looking-Glass

"Well, in our country," said Alice, still panting a little, "you'd generally get to somewhere else – if you run very fast for a long time, as we've been doing."

"A slow sort of country!" said the Queen. "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"

- Klein – Gordon equation on curved spacetime ($\varphi(x)$ real)

$$\partial_\nu \left(\sqrt{|g|}\, g^{\mu\nu} \partial_\mu \varphi\right) + \mu^2 \sqrt{|g|}\, \varphi = 0$$

i.e.

$$(\Box_g + \mu^2)\varphi = 0.$$

- Conjugate momentum

$$\pi(x) = \sqrt{|g|}\, g^{\mu 0} \partial_\mu \varphi(x)$$

- Phase space $(x = (x^0, \underline{x}))$

$$\Gamma = \left\{ (\varphi(\underline{x}), \pi(\underline{x}))_{\underline{x} \in \mathbb{R}^3} \right\}$$

with Poisson brackets

$$\{\pi(\underline{x}), \varphi(\underline{y})\} = \delta^{(3)}(\underline{x} - \underline{y}), \quad \{\pi(\underline{x}), \pi(\underline{y})\} = \{\varphi(\underline{x}), \varphi(\underline{y})\} = 0.$$

Let f, h be complex solutions of KG, let

$$j^\mu = i g^{\mu\nu} (\bar{f}\, \partial_\nu h - (\partial_\nu \bar{f}) h).$$

Then $j^\mu{}_{;\mu} = 0$

Pf.

$$j^\mu{}_{;\mu} \sqrt{|g|} = (\sqrt{|g|}\, j^\mu)_{,\mu}$$

$$= i \Big(\partial_\mu \bar{f} \cdot \sqrt{|g|}\, g^{\mu\nu} \partial_\nu h + \bar{f} \cdot \overbrace{\partial_\mu (\sqrt{|g|}\, g^{\mu\nu} \partial_\nu h)}^{-\mu^2 h \sqrt{|g|}}$$

$$\underbrace{- \partial_\mu (\sqrt{|g|}\, g^{\mu\nu} \partial_\nu \bar{f})}_{\mu^2 \bar{f} \sqrt{|g|}} \cdot h - \sqrt{|g|}\, g^{\mu\nu} \partial_\nu \bar{f} \cdot \partial_\mu h \Big)$$

$$= 0$$

Quantization

- Γ: phase space
  $\mathcal{F}(\Gamma) = \{\text{functions } a : \Gamma \to \mathbb{C}\}$
  (classical observables)
- $\mathcal{A}$ algebra with involution $* : A \mapsto A^*$
  (quantum observables)
- Quantization: map

  $$\mathcal{F}(\Gamma) \to \mathcal{A} \quad , \quad a \mapsto A$$

  with
  - $\bar{a} \mapsto A^*$
  - for some canonical coordinates $a, b, \dots$

    $$\{a, b\} \mapsto i[A, B]$$

- States on $\mathcal{A}$: Linear maps

  $$\omega : \mathcal{A} \to \mathbb{C} \quad , \quad A \mapsto \omega(A)$$

  with
  - $\omega(1) = 1 \quad , \quad \omega(A^*A) \geq 0$
  - meaning: $\omega(A)$ is the expectation of A in the state ω.

Conseq.: $|\omega(A^*B)|^2 \leq \omega(A^*A)\,\omega(B^*B)$
(consider $\omega((\bar{\lambda}A^* + \bar{\mu}B^*)(\lambda A + \mu B)) \geq 0$).

• Hilbert space: constructed for a given state ω by the GNS construction:

<u>Theorem</u>. Let ω be a state on $\mathcal{A}$. Then there are

- a Hilbert space $\mathcal{H}$
- a vector $\Omega \in \mathcal{H}$
- a representation π of $\mathcal{A}$ on $\mathcal{H}$

such that

$$\omega(A) = (\Omega, \pi(A)\Omega)$$

and $\{\pi(A)\Omega \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

Remark: Any normalized vector $\psi \in \mathcal{H}$ defines a state $A \mapsto (\psi, \pi(A)\psi)$; same for any density matrix P on $\mathcal{H}$: $A \mapsto \mathrm{tr}(PA)$.

But: for given ω not all other states are obtained this way (for systems of ∞ many degrees of freedom).

- Particle/antiparticle split of Klein-Gordon solutions

$$\mathcal{K} = \mathcal{H} \oplus \bar{\mathcal{H}}$$

with $\bar{\mathcal{H}} = C\mathcal{H}$ (C: complex conjugation)

$$\langle f, f \rangle \geq 0 \qquad (\forall f \in \mathcal{H})$$

$$\langle f, h \rangle = 0 \qquad (\forall f \in \mathcal{H}, \forall h \in \bar{\mathcal{H}})$$

- States ω on $\mathcal{A}$

$$\omega(a^*(f)\,a(h)) = \langle h, \rho f \rangle$$

respecting the split:

$$\rho = N \oplus (-1 - \bar{N})$$

with

$$\langle f, Nf \rangle \geq 0 \qquad (\forall f \in \mathcal{H})$$

$$\bar{N} = CNC$$

Quantization of Klein-Gordon eq. in Minkowski space

$\mathcal{H} \subset K$ particle subspace

$= \{ f(x)$ is superposition of positive frequency solutions$\}$

States $\rho = N \oplus (-1-\bar{N})$ w.r.t. $K = \mathcal{H} \oplus \bar{\mathcal{H}}$

- $N = 0$: Minkowski vacuum

$$\omega(a^*(f)a(h)) = 0 \qquad (f, h \in \mathcal{H})$$

in particular: expected number of particles in 1-particle state $f \in \mathcal{H}$

$$\omega(a^*(f)\, a(f)) = 0$$

- $N = \dfrac{1}{e^{\beta\omega(\vec{k})} - 1}$: thermal state

in momentum space.

For wave packet centered at $\vec{k}_0$:

$$\omega(a^*(f)\, a(f)) = \frac{1}{e^{\beta\omega(\vec{k}_0)} - 1}$$

Regge-Wheeler "tortoise" coordinates for Schwarzschild

Transition function

$$(t, r, \theta, \varphi) \longmapsto (t, r_*, \theta, \varphi)$$

Schwarzschild RW

given as

t, θ, φ unchanged

$$r_* = r + 2m \log\left(\frac{r}{2m} - 1\right).$$

coordinate intervals

$$r \in (2m, +\infty) \longmapsto r_* \in (-\infty, \infty)$$

Since

$$\frac{dr_*}{dr} = 1 + \left(\frac{r}{2m} - 1\right)^{-1} = \left(1 - \frac{2m}{r}\right)^{-1}$$

<u>metric</u> is

$$ds^2 = \left(1 - \frac{2m}{r}\right)(dt^2 - dr_*^2) + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

with $r = r(r_*)$.

Geodesics in the Schwarzschild metric
(time-like : $\mathcal{L} = -1$; $\dot{} = d/d\tau$)

- conservation laws

$$r^2 \dot{\varphi} = \ell \qquad \text{(angular momentum)}$$

$$\left(1 - \frac{2m}{r}\right) \dot{t} = \mathcal{E} \qquad \text{(energy)}$$

- radial equation

$$\dot{r}^2 + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) = \mathcal{E}^2$$

Radially infalling particle crossing horizon $r = 2m$, $t = +\infty$ at $\tau = 0$:

$$\dot{r} \cong -\mathcal{E} \quad , \quad \frac{r-2m}{2m} \dot{t} = \mathcal{E}$$

Thus

$$r - 2m = -\mathcal{E}\tau \quad , \quad \dot{t} = -\frac{2m}{\tau}$$

$$t = -2m \log(-\tau) + \text{const}$$

$$r_* = r + 2m \log\left(\frac{r}{2m} - 1\right)$$

$$= 2m + 2m \log\left(-\frac{\mathcal{E}\tau}{2m}\right)$$

Klein-Gordon eq. in RW coordinates

- separation of angular part

$$f(t, r_*, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{f_{\ell m}(t, r_*)}{r} Y_{\ell m}(\theta, \varphi)$$

- radial KG eq.

$$(\partial_t^2 - \partial_{r_*}^2 + V_{\ell m}) f_{\ell m} = 0$$

with effective potential

$$V_{\ell m}(r) = \left(1 - \frac{2m}{r}\right)\left(\frac{2m}{r^3} + \frac{\ell(\ell+1)}{r^2} + \mu^2\right)$$

- limits

$$V_{\ell m}(r) \to \begin{cases} 0 & r_* \to -\infty, \ (r \downarrow 2m) \\ \mu^2 & r_* \to +\infty, \ (r \to +\infty) \end{cases}$$

- solutions near $r_* \to -\infty$

$$f_{\ell m}(t, r_*) = f_{in}(t - r_*) + f_{out}(t + r_*)$$

$$\left.\begin{array}{l} \underline{\text{in}}\text{coming to} \\ \underline{\text{out}}\text{going from} \end{array}\right\} \text{exterior region}$$

Nearly flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

perturbation

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad , \quad |h_{\mu\nu}| \ll 1$$

In linear approximation in h

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu,\nu} + h^{\alpha}_{\nu,\mu} - h_{\mu\nu}{}^{,\alpha}\right)$$

(raising & lowering of indices by $\eta_{\mu\nu}$)

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}_{\nu\mu,\beta} - \Gamma^{\alpha}_{\beta\mu,\nu}$$

$$R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} - h_{,\mu\nu} + h^{\alpha}_{\mu,\alpha\nu} + h^{\alpha}_{\nu,\alpha\mu}\right)$$

with $h = h^{\alpha}{}_{\alpha}$, $\Box = \partial^{\alpha}\partial_{\alpha}$

Field equations ($G_{\mu\nu} = \kappa T_{\mu\nu}$) linearized

$$-\Box\gamma_{\mu\nu} - \eta_{\mu\nu}\gamma^{\alpha\beta}{}_{,\alpha\beta} + \gamma^{\alpha}_{\mu,\alpha\nu} + \gamma^{\alpha}_{\nu,\alpha\mu} = 2\kappa T_{\mu\nu}$$

with "trace reversed" perturbation

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Gauge transformations

$$h_{\mu\nu} \longmapsto h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$$

resp.

$$\gamma_{\mu\nu} \longmapsto \gamma_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu}\, \xi^{\alpha}{}_{,\alpha}$$

Gauges

i) Hilbert gauge : $\gamma^{\mu\nu}{}_{,\nu} = 0$

Lin. Field Eqs. $-\Box \gamma^{\mu\nu} = 2\varkappa T^{\mu\nu}$

ii) (in vacuum) Transverse Traceless gauge

$$h^{\mu 0} = 0 \,, \quad h^{i}{}_{i} = 0 \,, \quad h^{ij}{}_{,j} = 0$$

Lin. Field Eqs: $\quad \Box h_{ij} = 0$

Plane wave solutions : $\quad x = (t, \vec{x})$

$$h_{ij}(x) = h_{ij}(\underbrace{\vec{e}\cdot\vec{x} - t}_{s}) \,, \quad (|\vec{e}| = 1)$$

with

$$\frac{dh_{ij}}{ds} \cdot e^{j} = 0$$

Assuming $h_{ij}(s) \rightarrow 0 \; (s \rightarrow +\infty)$:

$$h_{ij}(s)\, e^{j} = 0$$

<u>Motion</u> of test particles in a plane wave

Particles initially "at rest" w.r.t. TT coordinates remain so:

$$x^\mu(\tau) = (\tau, \underbrace{\vec{x}_0}_{\text{fixed}})$$

Hence fixed coordinate differences between nearby particles

$$n^\mu(\tau) = (0, \underbrace{\vec{n}}_{\text{fixed}})$$

Distance Δs changes:

$$-\Delta s^2 = (n, n) = \underbrace{g_{\mu\nu}}_{\eta_{\mu\nu} + h_{\mu\nu}} n^\mu n^\nu$$

$$= -\vec{n}^2 + h_{ij}(s)\, n^i n^j$$

Electromagnetic, monochromatic plane wave

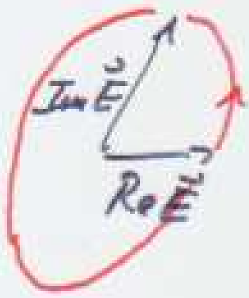
$(c=1)$

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left(\vec{E}\, e^{i\omega(\vec{e}\vec{x}-t)}\right), \quad \vec{E} \perp \vec{e}$$

$\uparrow$ complex

polarization

$$\vec{E}(0,t) = (\mathrm{Re}\,\vec{E})\cos\omega t + (\mathrm{Im}\,\vec{E})\sin\omega t$$



ellipse

special polarizations

i) $\mathrm{Re}\,\vec{E} \parallel \mathrm{Im}\,\vec{E}$ linear polarization

ii) $\mathrm{Re}\,\vec{E} \perp \mathrm{Im}\,\vec{E}$,
same length

$$\mathrm{Im}\,\vec{E} = \pm R_{\pi/2}(\mathrm{Re}\,\vec{E})$$

rotation by $\pi/2$

$$\vec{E} = A\begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \quad R_\varphi \vec{E} = e^{\pm i\varphi}\vec{E}$$

Electromagnetic dipole radiation $(c=1)$

$$A^{\mu}(\vec{x},t) = \frac{1}{4\pi r} \underbrace{\int d^3y \; j^{\mu}(\vec{y}, t-r)}$$

for $\mu = i$:

$$= \frac{d}{dt} \underbrace{\int d^3y \; \rho(\vec{y},t)\, y^i}_{= p^i \text{ : dipole moment}}$$