General relativity, solution sheet 12.

HS 08

1. Tidal forces near a black hole

i) For $(n, R(u,n)n)$ to describe the relative displacement of nearby geodesics, the vector field n needs to be Lie transported along u. For $n = \partial/\partial r$ and $u = 0$ initially, $(n,u) = 0$. Since this property is not conserved, n is not Lie transported and this choice is wrong.

ii) At a fixed proper time slice $\{\tau = 0\}$, let us choose a vector $n = (n^0, n^1, 0, 0)$ such that $(u,n) = 0$. It is unique up to normalization. n can be extended to a vector field on the whole manifold by Lie transport along u. Since $u = (u^0(t,r), u^1(t,r), 0, 0)$, the matrix of partial derivatives of its induced flow φ_τ is

$$D\varphi_\tau = \left(\begin{array}{c|c} M_\tau & 0_2 \\ \hline 0_2 & 1_2 \end{array}\right),$$

where M is a 2×2 matrix. This proves that n stays of the same form everywhere. Hence,

$$\frac{(n, R(u,n)u)}{(n,n)} = \frac{2m}{r^3}\frac{-(n^0u^1 - u^0n^1)^2}{(n,n)} = \frac{2m(u,u)(n,n)}{r^3(n,n)} = \frac{2m}{r^3}$$

by the equation given in the hint for the second equality, and $(u,u) = 1$.

iii) We choose $n = (0,0,n^2,0)$ at $\{\tau = 0\}$ which trivially satisfies $(u,n) = 0$ and extend it everywhere through Lie transport. By the same argument as before, it is also of that form everywhere. The tidal force in the 2-direction

$$\begin{aligned}\frac{(n, R(u,n)u)}{(n,n)} &= \frac{m}{r^3}\cdot\frac{(n^0u^2 - u^0n^2)^2 - (n^1u^2 - u^1n^2)^2}{(n,n)} = \frac{m}{r^3}\cdot\frac{((u^0)^2 - (u^1)^2)(n^2)^2}{(n,n)} \\ &= -\frac{m(u,u)(n,n)}{r^3(n,n)} = -\frac{m}{r^3}.\end{aligned}$$

A similar argument for $n = (0,0,0,n^3)$ produces the same answer. In particular, the average over all directions vanishes indeed.

2. Surface gravity

In Schwarzschild coordinates $u = (u^t, \vec{0}\,)$ with $u^t = (g_{tt})^{-1/2} = u^t(r)$, so that

$$(a,a) = g_{\mu\nu}(u^\mu{}_{,t} + \Gamma^\mu{}_{tt}u^t)(u^\nu{}_{,t} + \Gamma^\nu{}_{tt}u^t)(u^t)^2 = g_{rr}(\Gamma^r{}_{tt})^2(u^t)^4 = \frac{(g_{tt,r})^2(g_{tt})^{-2}}{4g_{rr}} = -\frac{(g_{tt,r})^2}{4g_{tt}},$$

where we used that $\Gamma^r{}_{tt} = -(1/2)(g_{rr})^{-1}g_{tt,r}$ and $g_{tt} = -(g_{rr})^{-1}$. Finally, since $\partial\tau/\partial t = (g_{tt})^{1/2}$, the surface gravity is given by

$$\kappa = \lim_{r\to 2m}\frac{g_{tt,r}}{2} = \lim_{r\to 2m}\frac{m}{r^2} = \frac{1}{4m}.$$