General relativity, solution sheet 2.

HS 08

1. Jacobi identity

i)

$$\begin{aligned}&[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]\\&=[X,Y]Z-Z[X,Y]+[Y,Z]X-X[Y,Z]+[Z,X]Y-Y[Z,X]\\&=XYZ-YXZ-ZXY+ZYX+YZX-ZYX\\&\quad -XYZ+XZY+ZXY-XZY-YZX+YXZ=0\,.\end{aligned}$$

ii) For any α,β,γ,

$$0=C^{\mu}{}_{\alpha\beta}[Y_\mu,Y_\gamma]+C^{\mu}{}_{\beta\gamma}[Y_\mu,Y_\alpha]+C^{\mu}{}_{\gamma\alpha}[Y_\mu,Y_\beta]=(C^{\mu}{}_{\alpha\beta}C^{\nu}{}_{\mu\gamma}+C^{\mu}{}_{\beta\gamma}C^{\nu}{}_{\mu\alpha}+C^{\mu}{}_{\gamma\alpha}C^{\nu}{}_{\mu\beta})Y_\nu\,.$$

Thus, the function $C^{\mu}{}_{\alpha\beta}C^{\nu}{}_{\mu\gamma}+C^{\mu}{}_{\beta\gamma}C^{\nu}{}_{\mu\alpha}+C^{\mu}{}_{\gamma\alpha}C^{\nu}{}_{\mu\beta}$ must be identically zero by the linear independence of the basis vectors $\{Y_\nu\}_{\nu=1}^{n}$ at each point $p\in M$.

2. Lie groups and Lie brackets

i) a) Let $r(t)$ be a curve in $\mathrm{O}(n)$ such that $r(0)=e$, denote $\dot r(0)\equiv s$. Then

$$0=\frac{d}{dt}\,r(t)^T r(t)\bigg|_{t=0}=s^T+s\,,$$

so that $\mathrm{Lie}(\mathrm{O}(n))=\{s\in\mathfrak{gl}(n,\mathbb{R})\mid s^T=-s\}$.

b) Similarly for a curve $l(t)\in\mathrm{SO}(1,3)$ with $l(0)=e$ and $\dot l(0)\equiv j$,

$$0=\frac{d}{dt}\,l(t)^T\eta l(t)\bigg|_{t=0}=j^T\eta+\eta j\,.$$

Thus, $\mathrm{Lie}(\mathrm{SO}(1,3))=\{j\in\mathfrak{gl}(4,\mathbb{R})\mid \eta j\eta^{-1}=-j^T\}$.

ii) Let $m_i(t)\in G$ $(i=1,2)$, the two curves generated by x_i, i.e. $x_i=\dot m_i(0)$. The generator of the curve $m_1(\alpha_1 t)m_2(\alpha_2 t)$ is

$$\frac{d}{dt}(m_1(\alpha_1 t)m_2(\alpha_2 t))\bigg|_{t=0}=\alpha_1x_1+\alpha_2x_2\in\mathrm{Lie}(G)\,. \tag{3}$$

For any s, $m_1(t)m_2(s)m_1(t)^{-1}m_2(s)^{-1}$ is a curve through e. Hence

$$\frac{d}{dt}(m_1(t)m_2(s)m_1(t)^{-1}m_2(s)^{-1})\bigg|_{t=0}=x_1-m_2(s)x_1m_2^{-1}(s)\in\mathrm{Lie}(G)\,.$$

By the linearity just proven, the derivative of a Lie algebra valued function is again an element of the algebra, i.e.

$$\frac{d}{ds}(x_1-m_2(s)x_1m_2^{-1}(s))\bigg|_{s=0}=-x_2x_1+x_1x_2\in\mathrm{Lie}(G)\,. \tag{4}$$

Finally, (4) follows from (2,3) and the distributivity for $n \times n$ matrices.

iii) It follows from $\varphi_*[X,Y] = [\varphi_* X, \varphi_* Y]$, see lecture notes p.7, that

$$(\lambda_g)_*[X,Y] = [X,Y]$$

for any two left-invariant vector fields X, Y.

iv) Applying $(\lambda_g)_*$ to (1) gives

$$[Y_\alpha, Y_\beta] = (\lambda_g)_*({C^\gamma}_{\alpha\beta} Y_\gamma) = ({C^\gamma}_{\alpha\beta} \circ \lambda_g) Y_\gamma \,.$$

Comparison with (1) yields, by the independence of the Y_γ, ${C^\gamma}_{\alpha\beta} \circ \lambda_g = {C^\gamma}_{\alpha\beta}$, i.e. ${C^\gamma}_{\alpha\beta}(gh) = {C^\gamma}_{\alpha\beta}(h)$. Hence ${C^\gamma}_{\alpha\beta}$ is constant.

v) Left-invariance means $X_{gh} = (\lambda_g)_* X_h$. For $h = e$ this shows that $X_e \in T_e(G) = \mathrm{Lie}(G)$ determines X_g at all $g \in G$.

vi) Let us first recall that the relation $X \leftrightarrow x$ between the (abstract) left-invariant vector field X and its matrix form x is

$$(Xf)(e) = \frac{d}{dt} f(m(t))\Big|_{t=0} \,, \qquad (f \in \mathcal{F}(G)) \,,$$

where $\dot{m}(0) = x$. Hence $\alpha_1 x_1 + \alpha_2 x_2$ corresponds to (see (3))

$$\begin{aligned} \frac{d}{dt} f(m_1(\alpha_1 t) m_2(\alpha_2 t))\Big|_{t=0} &= \frac{d}{dt} f(m_1(\alpha_1 t))\Big|_{t=0} + \frac{d}{dt} f(m_2(\alpha_2 t))\Big|_{t=0} \\ &= \alpha_1 (X_1 f)(e) + \alpha_2 (X_2 f)(e) \,, \end{aligned}$$

by using the chain rule, i.e., to $\alpha_1 X_1 + \alpha_2 X_2$. To establish the second correspondence consider the conjugation $\tau_h : G \to G$, $g \mapsto hgh^{-1}$. One verifies $\lambda_g \circ \tau_h = \tau_h \circ \lambda_{h^{-1}gh}$, implying $(\lambda_g)_*(\tau_h)_* = (\tau_h)_*(\lambda_{h^{-1}gh})_*$. Hence $(\tau_{h*})X$ is left-invariant if X is; moreover $(\tau_{h*})X \leftrightarrow hxh^{-1}$ because

$$\frac{d}{dt} f(hm(t)h^{-1})\Big|_{t=0} = \frac{d}{dt} (f \circ \tau_h)(m(t))\Big|_{t=0} = (\tau_{h*} X f)(e) \qquad (5)$$

by the definition of the tangent map. In particular

$$(\tau_{m_2(s)})_* X_1 \longleftrightarrow m_2(s) x_1 m_2(s)^{-1}$$

and by the already established linearity of the correspondence it extends to the derivative w.r.t. s. On the l.h.s we have by (5)

$$\begin{aligned} \frac{d}{ds} \tau_{m_2(s)*} X_1 f(e)\Big|_{s=0} &= \frac{d}{ds}\frac{d}{dt} f(m_2(s) m_1(t) m_2(s)^{-1})\Big|_{s=t=0} \\ &= \frac{d}{ds}\frac{d}{dt} \big(f(m_2(s) m_1(t)) + f(m_1(t) m_2(s)^{-1})\big)\Big|_{s=t=0} \\ &= \frac{d}{ds} X_1 (f \circ \lambda_{m_2(s)})(e)\Big|_{s=0} - \frac{d}{dt} X_2 (f \circ \lambda_{m_1(t)})(e)\Big|_{t=0} \\ &= ((X_1 X_2 - X_2 X_1) f)(e) \,, \end{aligned}$$

i.e.

$$\frac{d}{ds}\tau_{m(s)*}X_1\bigg|_{s=0} = [X_1, X_2]\,.$$

On the r.h.s. we have, see (4)

$$\frac{d}{ds}m_2(s)x_1m_2(s)^{-1}\bigg|_{s=0} = [x_1, x_2]\,.$$