General relativity, solution sheet 8.

HS 08

1. Energy conditions

The dual basis is $e^0 = e_0$, $e^i = -e_i$. We have therefore $T_{00} = T^0{}_0 = T^{00}$, $T_{0i} = -T^{0i}$, $T_{ik} = -T^i{}_k = T^{ik}$ and in particular $T = T^\alpha{}_\alpha = T_{00} - \sum_{i=1}^3 T_{ii}$.

i) T_{00} in the rest frame of an observer with 4-velocity u is written in general covariant form as $T_{00} = (Tu, u)$, because in his rest frame $u = e_0$, i.e. $(Tu, u) = T(e_0, e_0) = T_{00}$. The weak energy condition is then $(Tu, u) \geq 0$ for every timelike vector u.

ii) $(T_{\alpha\beta} - (1/2)g_{\alpha\beta}T)u^\alpha u^\beta$ is invariant and in the rest frame of the observer it is equal to

$$T_{00} - \frac{1}{2}\eta_{00}T = \frac{1}{2}(T_{00} + \sum_{i=1}^{3} T_{ii}) \, .$$

The strong energy condition is therefore $(T_{\alpha\beta} - (1/2)g_{\alpha\beta}T)u^\alpha u^\beta \geq 0$ for every timelike vector u. According to the calculations done at page 36 we have $\nabla^2_{e_0} n_a|_{\tau=0} = R(e_0, e_a)e_0$, i.e.

$$(n_a, \nabla^2_{e_0} n_a)\big|_{\tau=0} = -\langle e^a, \nabla^2_{e_0} n_a|_{\tau=0}\rangle = \langle e^a, R(e_a, e_0)e_0\rangle = R^a{}_{0a0}$$

and, as $R^0{}_{000} = 0$,

$$\sum_{a=1}^{3} (n_a, \nabla^2_{e_0} n_a)\big|_{\tau=0} = R^\alpha{}_{0\alpha0} = R_{00} = \kappa(T_{00} - \frac{1}{2}\eta_{00}T) \geq 0 \, .$$

iii) The cone of timelike or lightlike vectors is $\bar{V}^+ := \{v \mid (v, v) \geq 0,\, v^0 \geq 0\}$. In the rest frame of the observer we have $(Tu)^\alpha = T^{\alpha\beta}u_\beta = T^{\alpha 0}$, therefore the dominant energy condition is equivalent to

$$(Tu, u) \geq 0 \, , \qquad (Tu, Tu) \geq 0$$

for every timelike vector u.

We note that $v \in \bar{V}^+$ iff $(v, w) \geq 0$ for all $w \in \bar{V}^+$ (such vectors are of the form $(v^0, \vec{v})$ with $v^0 \geq |\vec{v}|$, therefore $(v, w) = v^0 w^0 - \vec{v} \cdot \vec{w} \geq 0$ by $|\vec{v}||\vec{w}| \geq |\vec{v} \cdot \vec{w}|$). Applying this to $v = Tu$, the dominant energy condition states $(w, Tu) \geq 0$ for any timelike vectors u, w. Taking them with $u^0 = w^0 = 1$, and hence $|\vec{u}|, |\vec{w}| \leq 1$, yields

$$(w, Tu) = T_{00} + T_{0k}(u^k + w^k) + w^i T_{ik} u^k \geq 0 \, ,$$

($T_{k0} = T_{0k}$). In particular for $\vec{u} = \vec{w} = 0$ the weak energy condition follows; picking $\vec{u} = \pm\vec{e_i}$, $\vec{w} = 0$ implies $T_{00} \pm T_{0i} \geq 0$, i.e. $|T_{0i}| \leq T_{00}$; by setting instead $\vec{w} = \pm\vec{e_k}$ we get

$$T_{00} \pm (T_{0i} + T_{0k}) + T_{ik} \geq 0 \, ,$$

and $T_{00} + T_{ik} \geq 0$ by the sum of the two inequalities. In the same way $T_{00} - T_{ik} \geq 0$ (and therefore the claim) follows from the choice $\vec{w} = \mp\vec{e_k}$. Conversely: by a rotation of

e_a, $(a = 1, 2, 3)$ with e_0 fixed, T^{00} does not change and $(T^{a0})_{a=1}^3$ transforms as a Euclidean vector, with no consequences on the property $(T^{\mu 0})_{\mu=0}^3 \in \bar{V}^+$; we may therefore assume $T^{10} = T^{20} = 0$ and the dominant energy condition follows then from $|T^{30}| \le T^{00}$.

iv) As a preliminary we consider tensors of the form $T = au \otimes u - b(u, u)g$ with $(u, u) > 0$. We have

$$(v, Tv) = a(v, u)^2 - b(u, u)(v, v) = a\big((v, u)^2 - (u, u)(v, v)\big) + (a - b)(u, u)(v, v) .$$

In both terms the factors containing v are ≥ 0 for $(v, v) \ge 0$. Indeed, $\vec{u} = 0$ without loss by invariance; then the first factor is $(u^0)^2 = (u, u)$ times $(v^0)^2 - (v, v) = \vec{v}^2$. The two terms can however be made $= 0$ independently of one another. Thus $(v, Tv) \ge 0$ for all $v \in \bar{V}^+$ iff $a - b \ge 0$ and $a \ge 0$.

Let us put $c = 1$. For the ideal fluid, $a = \rho + p$ and $b = p$. The condition (i) thus implies $\rho \ge 0$ and $\rho + p \ge 0$. Since $\operatorname{tr} T = \rho - 3p$ the tensor $T - (1/2)(\operatorname{tr} T)g$ is of the same form, with the same a but with $b = p + (1/2)(\rho - 3p) = (1/2)(\rho - p)$. Thus (ii) amounts to $\rho + 3p \ge 0$ and $\rho + p \ge 0$. Finally $(T^2)^{\mu\nu} := T^\mu{}_\sigma T^{\sigma\nu}$ is still of the same form:

$$T^2 = (a^2(u, u)u \otimes u - 2ab(u, u)u \otimes u + b^2(u, u)^2 g = (u, u)\big(a' u \otimes u - b'(u, u)g\big)$$

with $a' = a^2 - 2ab$, $b' = -b^2$. The condition $0 \le a' - b' = (a - b)^2$ is trivial and (iii) asks for $0 \le a' = \rho^2 - p^2$. In view of $\rho \ge 0$, this is $|p| \le \rho$. One can reinstate c by $p \rightsquigarrow pc^{-2}$.

Alternatively the same conclusions may be obtained from the formulations of the energy conditions in terms of matrix elements. We have

$$T^{00} = (\rho + p)(u^0)^2 - p = (\rho + p)((u^0)^2 - 1) + \rho ,$$

$$T_{00} + \sum_{i=1}^{3} T_{ii} = (\rho + p)((u^0)^2 + \vec{u}^2) - p + 3p = 2(\rho + p)((u^0)^2 - 1) + \rho + 3p$$

by using $(u^0)^2 - \vec{u}^2 = 1$. Both expression are linear in $(u^0)^2 \in [1, \infty)$, hence the conclusion about (i,ii). The condition (iii), i.e. $|\vec{T}| \le T^{00}$ for $\vec{T} = (T^{i0})$, is equivalent to $T^{00} \ge 0$ and $|\vec{T}|^2 \le (T^{00})^2$. The latter inequality is, after some rearrangement, $0 \le (\rho^2 - p^2)((u^0)^2 - 1) + \rho^2$, whence the conclusion.

For the electromagnetic field we have $T^{00} = (1/2)(\vec{E}^2 + \vec{B}^2) \ge 0$. Therefore (i,ii) are satisfied (the difference between the two disappears because of $\operatorname{tr} T = 0$). Also (iii) is satisfied because $T^{0i} = (\vec{E} \wedge \vec{B})^i$ and $|\vec{E} \wedge \vec{B}| \le |\vec{E}||\vec{B}| \le T_{00}$.

For the cosmological term $T^{\mu\nu} = (\Lambda/\kappa)g^{\mu\nu}$ with $\Lambda > 0$ we have: (i) is satisfied, (ii) is not $(T^{\mu\nu} - (1/2)Tg^{\mu\nu} = -(\Lambda/\kappa)g^{\mu\nu})$ and (iii) again is. Less directly, this also follows from the fluid with $p = -\rho$.

The strong energy condition plays a role in Hawking's singularity theorem described below.

Let (M, g) be a pseudo-Riemannian manifold of signature $(+, -, -, -)$ and $\Sigma \subset M$ be a spacelike 3-surface having normal u: $g(u, u) = 1$, $g(u, X) = 0$ for any vector field X on Σ. Conventionally call the side of Σ distinguished by u its future side. Define a (symmetric) tensor K of type $\binom{0}{2}$ on Σ, called extinsic curvature, by

$$K(X, Y) = g(\nabla_X u, Y) ,$$

where X, Y are vector fields on Σ.

Theorem (roughly). Suppose Σ is compact and $\operatorname{tr} K \le C < 0$ on Σ. Suppose

$$R_{\mu\nu}\xi^\mu\xi^\nu \ge 0 \tag{1}$$

for any timelike vector field ξ on M. Then there is a timelike geodesic starting from the past side of Σ, such that it ends in a singularity of M. In fact, it reaches it within proper time $3/|C|$.

Remarks. 1) For g a solution of the field equations (5.9), eq. (1) amounts to the strong energy condition.

2) Example: Let Σ be a time slice in a Friedmann model (it is compact for $k = +1$). Then $u = (1, 0, 0, 0)$ in chart A on p. 41 of the lecture notes. At $(t, 0, 0, 0)$,

$$K_{ij} = a^2 \Gamma^j{}_{i0} = \delta_{ij} a\dot{a}\,.$$

In particular, $\operatorname{tr} K = -3\dot{a}a^{-1} < 0$ during expansion. Indeed, there is a singularity (the big bang) in the past of Σ, as the theorem claims. The theorem however shows that homogeneity and isotropy are not essential for that.