# General relativity, solution sheet 9.

---

*HS 08*

**1. Radiation dominated universe**

(i) At the point $(t,0,0,0)$, the metric and the energy-momentum tensor are

$$g_{\mu\nu} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -a^2\delta_{ij} \end{array}\right), \qquad T^{\mu\nu} = \left(\begin{array}{c|c} \rho & 0 \\ \hline 0 & a^{-2}\frac{\rho}{3}\delta_{ij} \end{array}\right)$$

The components of the Einstein tensor are still given by (6.10). Thus, the Friedmann equations are

$$\begin{aligned} (00):& \qquad a(\dot{a}^2+k) = \frac{1}{3}\rho a^3 , \\ (jj):& \qquad 2a\ddot{a}+\dot{a}^2+k = -\frac{1}{3}\rho a^2 . \end{aligned}$$

Summing them, one obtains $2(a\ddot{a}+\dot{a}^2+k)=0$ implying the conservation of $\rho a^4$ since

$$\frac{d}{dt}\frac{1}{3}\rho a^4 = \frac{d}{dt}a^2(\dot{a}^2+k) = 2a\dot{a}(a\ddot{a}+\dot{a}^2+k) = 0 .$$

Note that the $0-$component of integrability condition $T^{\mu\nu}{}_{;\nu}=0$ also implies this conservation law, since

$$0 = T^{0\nu}{}_{;\nu} = \frac{d\rho}{dt} + 4\frac{\dot{a}}{a}\rho = a^{-4}\frac{d}{dt}\rho a^4 .$$

(ii) With $\rho a^4/3 =: C^2$, the equation to be solved is

$$\dot{a}^2 + k - \frac{C^2}{a^2} = 0 , \qquad a(t=0) = 0 .$$

For $k=0$, the solution is

$$a(t) = (2Ct)^{1/2} .$$

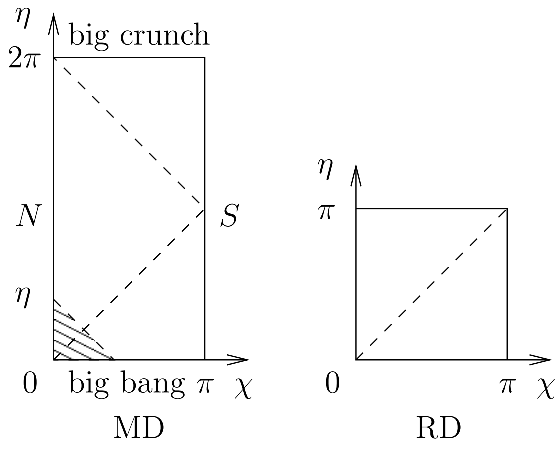
In the cases $k=\pm 1$, we use $\eta$ as independent variable, with $\eta(t=0)=0$ and $' = d/d\eta$. Since $t' = a$ and $\dot{a} = a'/t'$, the differential equation reads $a'^2 - C^2 + ka^2 = 0$, with solutions

$$\begin{aligned} &(k=1) \qquad a(\eta) = C\sin\eta , \quad t(\eta) = C(1-\cos\eta) , \\ &(k=-1) \qquad a(\eta) = C\sinh\eta , \quad t(\eta) = C(\cosh\eta - 1) . \end{aligned} \tag{2}$$

**2. The causal structure of the Friedmann models**

(i) For MD, $a(\eta) \propto 1-\cos\eta$, so that $\eta_0 = 2\pi$. For RD, (2) above implies $\eta_0 = \pi$.

(ii) The lines $\chi = 0$, resp. $\chi = \pi$, correspond to the north, resp. south, poles of the 3−sphere. The geodesics starting at $\chi = 0$ being radial ($d\theta = d\varphi = 0$), the metric $g = a^2(d\eta^2 - d\chi^2)$ along them is conformally equivalent to the Minkowski metric, so that light signals propagate at $\pm 45°$ in the $(\eta, \chi)$-plane.




In a MD universe, the light signal emitted by an observer (at the northpole) can only come back to him (over the south pole) if it can propagate throughout the whole lifetime of the universe. In a RD universe, this is not possible, the signal can only cover half of the distance.