General relativity, solution sheet 6.

HS 08

1. An ideal fluid

Energy density and pressure can be obtained from the 4-momentum $(E/c, \vec{p}) = m\gamma(c, \vec{v})$, with $\gamma = (1 - v^2/c^2)^{-1/2}$:

$$\rho c^2 = nm\gamma c^2 \,,$$
$$p = nm\gamma\langle v_i^2\rangle = \frac{1}{3}nm\gamma v^2 \,,$$

where n is the particle density. In the two limiting cases, the energy-momentum tensor $T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu - pg^{\mu\nu}$ reduces to

$$\text{i)} \quad T^{\mu\nu} \xrightarrow{v\to 0} \rho u^\mu u^\nu \quad (\text{dust})\,,$$
$$\text{ii)} \quad T^{\mu\nu} \xrightarrow{v\to c} \frac{4}{3}\rho u^\mu u^\nu - \frac{1}{3}\rho c^2 g^{\mu\nu} \,.$$

In the second case, $T^\mu{}_\mu = 0$, which is consistent with the interpretation of a photon gas as radiation field: The electromagnetic energy-momentum tensor satisfies indeed $T^\mu{}_\mu = F^{\mu\sigma}F_{\sigma\mu} - (1/4)F_{\rho\sigma}F^{\sigma\rho}\delta^\mu{}_\mu = 0$.

2. A variational principle

Following the hint, we parametrize the curve by a new parameter λ such that its values λ_1, resp. λ_2 at spacetime points (1), resp. (2), remain fixed under variations. Let $x' = \mathrm{d}x/\mathrm{d}\lambda$. Since $c\,\mathrm{d}\tau/\mathrm{d}\lambda = (x', x')^{1/2}$, the variational principle reads

$$0 = \delta \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \left(c\,(x', x')^{1/2} + \frac{e}{mc}(x', A) \right) \equiv \delta \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda\,(L_1 + L_2)\,.$$

On the one hand,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L_1}{\partial x'^\mu} = \frac{c}{(x', x')^{1/2}}\left(g_{\nu\mu}x''^\nu + g_{\nu\mu,\rho}x'^\nu x'^\rho\right) - g_{\nu\mu}x'^\nu \left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^{-2}\frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2}\,,$$
$$\frac{\partial L_1}{\partial x^\mu} = \frac{c}{2(x', x')^{1/2}}\, g_{\nu\rho,\mu}x'^\nu x'^\rho \,.$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L_2}{\partial x'^\mu} = \frac{e}{mc}A_{\mu,\nu}x'^\nu \,,$$
$$\frac{\partial L_2}{\partial x^\mu} = \frac{e}{mc}A_{\nu,\mu}x'^\nu \,.$$

Thus, the Euler-Lagrange equations are

$$\frac{c}{(x', x')^{1/2}}\left(g_{\nu\mu}x''^\nu + g_{\rho\mu}\Gamma^\rho{}_{\nu\sigma}x'^\nu x'^\sigma\right) - g_{\nu\mu}x'^\nu\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^{-2}\frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2} = \frac{e}{mc}F_{\mu\rho}x'^\rho \,.$$

Reparametrizing the trajectory with the proper time τ, one obtains the equation of motion

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = \frac{e}{mc}F^{\mu\nu}\dot{x}_\nu \,.$$