General relativity, solution sheet 1.

HS 08

1. Stereographic projection

The coordinates (x, y) of a point $p = (x_1, x_2, x_3) \in U_+$ are

$$\mathbb{R}^2 \ni (x, y) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right)$$

and similarly for $(\bar{x}, \bar{y})$ of $p \in U_-$,

$$\mathbb{R}^2 \ni (\bar{x}, \bar{y}) = \left(\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3} \right) .$$

The transition function $(x, y) \mapsto (\bar{x}, \bar{y})$ is defined on the image of the overlap $U_+ \cap U_-$ of the two coordinate patches, i.e. on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Since

$$\frac{1 - x_3}{1 + x_3} = \frac{1}{x^2 + y^2} ,$$

it is given by

$$(\bar{x}, \bar{y}) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) .$$

2. Tensors

a) Let $\{e_1 \, \ldots \, e_n\}$ be a basis for V. A general tensor $T \in V \otimes V$ is of the form $T = T^{ij} e_i \otimes e_j$ with any $n \times n$ matrix (T^{ij}). If $T = v_1 \otimes v_2$, with $v_k = v^i_{(k)} e_i$, then $T^{ij} = v^i_{(1)} v^j_{(2)}$ is a rank 1 matrix since each column vector is a multiple of v_2. Thus it cannot be general.

b) Tensors $T \in V \otimes W^*$ and linear maps $\tilde{T} : W \mapsto V$ are brought in one-to-one correspondence through

$$\langle \nu, \tilde{T} w \rangle = T(\nu, w) \, , \quad (\nu \in V^* \, , w \in W) \, .$$

3. On the Lie derivative

It follows from the transformation law

$$\overline{R}^{\alpha}{}_{\beta} = R^{i}{}_{j} \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{\beta}}$$

that

$$\overline{R}^{\alpha}{}_{\beta,\gamma} = R^{i}{}_{j,k} \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} + R^{i}{}_{j} \left(\frac{\partial^2 \bar{x}^{\alpha}}{\partial x^{i} \partial x^{k}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} + \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial^2 x^{j}}{\partial \bar{x}^{\beta} \partial \bar{x}^{\gamma}} \right) ;$$

Similarly, from

$$\overline{X}^{\gamma} = X^{l} \frac{\partial \bar{x}^{\gamma}}{\partial x^{l}} ,$$

that

$$\overline{X}^{\gamma}{}_{,\delta} = X^{l}{}_{,m} \frac{\partial \bar{x}^{\gamma}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\delta}} + X^{l} \frac{\partial^2 \bar{x}^{\gamma}}{\partial x^{l} \partial x^{m}} \frac{\partial x^{m}}{\partial \bar{x}^{\delta}} .$$

Plugging this into

$$(\overline{L_X R})^\alpha{}_\beta = \overline{R}^\alpha{}_{\beta,\gamma}\overline{X}^\gamma - \overline{R}^\gamma{}_\beta \overline{X}^\alpha{}_{,\gamma} + \overline{R}^\alpha{}_\gamma \overline{X}^\gamma{}_{,\beta}\,,$$

one obtains, for the terms without second derivatives,

$$\begin{aligned}
&R^i{}_{j,k}\frac{\partial\bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^\beta}\frac{\partial x^k}{\partial\bar{x}^\gamma}X^l{}_{,m}\frac{\partial\bar{x}^\gamma}{\partial x^l}\frac{\partial x^m}{\partial\bar{x}^\delta} - R^k{}_j\frac{\partial\bar{x}^\gamma}{\partial x^k}\frac{\partial x^j}{\partial\bar{x}^\beta}X^i{}_{,m}\frac{\partial\bar{x}^\alpha}{\partial x^i}\frac{\partial x^m}{\partial\bar{x}^\gamma} + R^i{}_k\frac{\partial\bar{x}^\alpha}{\partial x^i}\frac{\partial x^k}{\partial\bar{x}^\gamma}X^l{}_{,j}\frac{\partial\bar{x}^\gamma}{\partial x^l}\frac{\partial x^j}{\partial\bar{x}^\beta}\\
&= (R^i{}_{j,k}X^k - R^k{}_j X^i{}_{,k} + R^i{}_k X^k{}_{,j})\frac{\partial\bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^\beta} = (L_X R)^i{}_j\frac{\partial\bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^\beta}\,,
\end{aligned}$$

where we used

$$\frac{\partial x^j}{\partial\bar{x}^\gamma}\frac{\partial\bar{x}^\gamma}{\partial x^l} = \delta^j{}_l\,. \tag{3}$$

The terms containing second derivatives

$$\begin{aligned}
R^i{}_j\Big(\frac{\partial^2\bar{x}^\alpha}{\partial x^i\partial x^k}\frac{\partial x^k}{\partial\bar{x}^\gamma}\frac{\partial x^j}{\partial\bar{x}^\beta} + \frac{\partial\bar{x}^\alpha}{\partial x^i}\frac{\partial^2 x^j}{\partial\bar{x}^\beta\partial\bar{x}^\gamma}\Big)X^l\frac{\partial\bar{x}^\gamma}{\partial x^l} &- R^i{}_j\frac{\partial\bar{x}^\gamma}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^\beta}X^l\frac{\partial^2\bar{x}^\alpha}{\partial x^l\partial x^m}\frac{\partial x^m}{\partial\bar{x}^\gamma}\\
&+R^i{}_j\frac{\partial\bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^\gamma}X^l\frac{\partial^2\bar{x}^\gamma}{\partial x^l\partial x^m}\frac{\partial x^m}{\partial\bar{x}^\beta}
\end{aligned}$$

vanish. Indeed, the first and third term cancel each other. The two others do as well because

$$\frac{\partial^2 x^j}{\partial\bar{x}^\beta\partial\bar{x}^\gamma}\frac{\partial\bar{x}^\gamma}{\partial x^l} + \frac{\partial^2\bar{x}^\gamma}{\partial x^l\partial x^m}\frac{\partial x^m}{\partial\bar{x}^\beta}\frac{\partial x^j}{\partial\bar{x}^\gamma} = 0\,,$$

which is seen by taking the derivative of (3) w.r.t. $\overline{x}^\beta$. Thus, the components of $L_X R$ (defined in a chart) transform like a tensor.