General relativity, solution sheet 5.

HS 08

1. A free fall

Besides O, let us introduce a new, freely falling reference frame O' that is at rest w.r.t. O when the first mass is being dropped. In O', the falling mass is at rest and the second one has an inertial trajectory of speed $u = g\Delta t_2$. After time t the distance between the two masses is ut (neglecting the small initial distance $\propto (\Delta t_2)^2$ that would cancel out in the final result anyway). The clock C_1, with speed $v = gt$ meets the two masses in a time interval $ut/gt = \Delta t_2$ w.r.t. O', independent of the time t. In O, the two events happen at the same location, so that the time interval there reads $\Delta t_1 = \sqrt{1-(v/c)^2}\Delta t_2$ according to special relativity. Since $v^2/2 = gh$, we finally obtain

$$\frac{\Delta t_1}{\Delta t_2} = \sqrt{1 - \frac{2gh}{c^2}} = 1 - \frac{gh}{c^2} + O(c^{-4})\,.$$

2. The Newton's equation as a geodesic equation

With t as the parameter to describe the curve $x(t) = (t, \vec{x}(t))$, Newton's equation reads

$$\ddot{x}^0 = 0\,, \qquad \ddot{x}^i + \partial_i\varphi(\vec{x})(\dot{x}^0)^2 = 0\,,$$

which is the geodesic equation with ${\Gamma^i}_{00} = \partial_i\varphi$ and all other Γ's vanishing. They indeed define a symmetric connection, ${\Gamma^i}_{kl} = {\Gamma^i}_{lk}$. However, if the corresponding components

$${R^i}_{0k0} = \varphi_{,ik} \quad \text{and} \quad {R^i}_{jk0} = 0$$

were those of a Riemann tensor derived from a metric g_{ij}, then

$$R_{i0k0} = g_{im}\varphi_{,mk}\,, \qquad R_{0ik0} = g_{0m}{R^m}_{ik0} = 0\,,$$

which is in contradiction with the symmetry property $R_{i0k0} = -R_{0ik0}$ of metric curvature tensors.

3. On the Riemann connection

First of all, let us note that $(\nabla^{(N)}_X Y)_p \in T_p(M)$, $(p \in N)$, is defined for vector fields X, Y that are only live on N. This follows from the relation between parallel transport and affine connection. What needs to be shown here is that (i) $P \circ \nabla^{(M)}$ is an affine connection on N, (ii) its torsion vanishes and (iii) $(P \circ \nabla^{(M)})g = 0$. Then (2) holds by uniqueness of the Riemann connection $\nabla^{(N)}$.

(i) The f-linearity of $P\nabla^{(M)}_X Y$ in X:

$$(P\nabla^{(M)}_{fX} Y)_p = P_p(f(p)\nabla^{(M)}_X Y)_p = f(p)P_p(\nabla^{(M)}_X Y)_p = f(p)(P\nabla^{(M)}_X Y)_p\,,$$

because P_p is linear. The product rule w.r.t. Y:

$$(P\nabla^{(M)}_X fY)_p = P_p((Xf)(p)Y_p + f(p)(\nabla^{(M)}_X Y)_p) = (Xf)(p)Y_p + f(p)(P\nabla^{(M)}_X Y)_p\,,$$

since $P_pY_p = Y_p$.

(ii) The torsion property is directly inherited: $P_p(\nabla^{(M)}_X Y)_p - P_p(\nabla^{(M)}_Y X)_p = P_p(\nabla^{(M)}_X Y - \nabla^{(M)}_Y X)_p = P_p([X, Y])_p$.

(iii) Since $\nabla_Z f = Zf$ for any connection ∇, we have

$$\begin{aligned}(P\nabla^{(M)}_Z g)(X, Y) &= Zg(X, Y) - g(P\nabla^{(M)}_Z X, Y) - g(X, P\nabla^{(M)}_Z Y)\\ &= Zg(X, Y) - g(\nabla^{(M)}_Z X, Y) - g(X, \nabla^{(M)}_Z Y) = (\nabla^{(M)}_Z g)(X, Y) = 0\,.\end{aligned}$$