General relativity, solution sheet 10.

HS 08

1. The magnitude-redshift relation

i) The energy traversing the surface at distance d from $\chi = 0$ in the time interval δt is

$$\delta E = \frac{h\nu_2}{\Delta\tau^{(2)}}\delta t$$

($\Delta\tau^{(i)}$ is the proper time interval between successive photons), the area of this surface is $a = 4\pi(R_0 \mathrm{sinn}\chi)^2$, and with $\nu_2 = \nu_1(1+z)^{-1}$, $\Delta\tau^{(2)} = \Delta\tau^{(1)}(1+z)$, $d = \chi R_0$ we have

$$f = \frac{L}{4\pi d^2(1+z)^2}\big(\frac{\chi}{\mathrm{sinn}\chi}\big)^2 = \frac{L}{4\pi d^2(1+z)^2}\big(1 - \frac{\Omega_k}{3}(Hd)^2 + \ldots\big)$$

(the second equation follows from $\mathrm{sinn}\chi = \chi - (1/6)k\chi^3 + O(\chi^5)$ and (6.23)).

ii) Inverting the redshift-distance relation

$$z = Hd + \frac{1}{2}(1+q)(Hd)^2 + \ldots$$

we get

$$d = H^{-1}(z - \frac{1}{2}(1+q)z^2 + \ldots)\,,$$

and therefore

$$d^2(1+z)^2 = H^{-2}z^2(1 + (1-q)z + \ldots)\,,$$

so that

$$f = \frac{LH^2}{4\pi z^2}\big(1 - (1-q)z + O(z^2)\big)\,.$$

2. Killing vectors

i) The Killing equation $L_K g = 0$ is, in Minkowski coordinates,

$$K_{\mu,\nu} + K_{\nu,\mu} = 0\,. \tag{2}$$

This implies $\partial_\mu K_\mu = 0$ (without summation) and, by differentiation of (2),

$$\partial_\mu\partial_\mu K_\nu = -\partial_\nu\partial_\mu K_\mu = 0\,.$$

Every Killing vector is therefore of the form $K_\mu = a_\mu + b_\mu{}^\nu x_\nu$, with (2) implying that $b_{\mu\nu}$ is antisymmetric. We have then 10 independent Killing fields:

$$K^\mu_{(1)} = (1,0,0,0) \qquad K^\mu_{(2)} = (0,1,0,0) \qquad K^\mu_{(3)} = (0,0,1,0) \qquad K^\mu_{(4)} = (0,0,0,1)\,,$$

(generators of translations), and

$$\begin{array}{llll} K^\mu_{(5)} = (x,t,0,0) & K^\mu_{(6)} = (y,0,t,0) & K^\mu_{(7)} = (z,0,0,t) & \text{(generators of boosts)} \\ K^\mu_{(8)} = (0,y,-x,0) & K^\mu_{(9)} = (0,0,z,-y) & K^\mu_{(10)} = (0,-z,0,x) & \text{(generators of rotations)} \end{array}$$

where we have raised the indices.

ii) The Killing equation $K_{\mu;\nu} + K_{\nu;\mu} = 0$ is

$$K_{\mu,\nu} + K_{\nu,\mu} = 2\Gamma^{\lambda}{}_{\mu\nu} K_\lambda \,, \tag{3}$$

and the only nonvanishing Christoffel symbols are

$$\Gamma^{v}{}_{xx} = a(u)a'(u) \qquad \Gamma^{v}{}_{yy} = b(u)b'(u) \qquad \Gamma^{x}{}_{ux} = \frac{a'(u)}{a(u)} \qquad \Gamma^{y}{}_{uy} = \frac{b'(u)}{b(u)}\,.$$

The metric components g_{ij} do not depend on (v, y, z), we therefore have the three Killing fields

$$K^{\mu}_{(1)} = (0,1,0,0) \qquad K^{\mu}_{(2)} = (0,0,1,0) \qquad K^{\mu}_{(3)} = (0,0,0,1)\,.$$

To find the remaining two Killing fields we put the ansatz proposed on the exercise sheet $K^\mu = (0, f(u,x), g(u,x), h(u,x))$, i.e.

$$K_\mu = (f(u,x), 0, -a^2(u)g(u,x), -b^2(u)h(u,x))\,,$$

into the Killing equations (3) to get

$$\begin{pmatrix} 2\partial_u f & 0 & -\partial_u(a^2 g) + \partial_x f & \partial_u(b^2 h) \\ & 0 & 0 & 0 \\ & & -2a^2\partial_x g & -b^2\partial_x h \\ & & & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 0 & -aa'g & -bb'h \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix},$$

which implies $f = f(x)$, $g = g(u)$, $h = 0$ and

$$a^2(u)\partial_u g(u) = \partial_x f(x)\,,$$

with solution proportional to $f = x$, $g = \int a^{-2} du$. We have then $K^{\mu}_{(4)} = (0, x, \int a^{-2}du, 0)$; $K^{\mu}_{(5)} = (0, y, 0, \int b^{-2}du)$ is found in a similar way by means of the ansatz $K^\mu = (0, f(u,y), g(u,y), h(u,y))$.