General relativity, solution sheet 3.

HS 08

1. Parallel transport in polar coordinates

The coordinate transformation is $x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$, and we have

$$dx^1 = \cos\varphi\, dr - r\sin\varphi\, d\varphi\ , \qquad dx^2 = \sin\varphi\, dr + r\cos\varphi\, d\varphi\ .$$

Let (v^1, v^2) be the cartesian components of a vector, and (v^r, v^φ) the ones w.r.t. the basis $\{\partial/\partial r, \partial/\partial\varphi\}$. As (v^1, v^2) transform in the same way as (dx^1, dx^2) does, we have

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \begin{pmatrix} v^r \\ v^\varphi \end{pmatrix}, \quad \begin{pmatrix} v^r \\ v^\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r^{-1}\sin\varphi & r^{-1}\cos\varphi \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}.$$

We have defined the parallel transport along the curve $\gamma(r(t), \varphi(t))$ by requiring that the cartesian components remain constant, which implies for v_r, v_φ ($\dot{} = d/dt$):

$$\begin{aligned} \begin{pmatrix} \dot v^r \\ \dot v^\varphi \end{pmatrix} &= \left[\dot\varphi \begin{pmatrix} -\sin\varphi & r\cos\varphi \\ -r^{-1}\cos\varphi & -r^{-1}\sin\varphi \end{pmatrix} - \frac{\dot r}{r^2}\begin{pmatrix} 0 & 0 \\ -\sin\varphi & \cos\varphi \end{pmatrix}\right] \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \\ &= \left[\dot\varphi \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix} + \dot r \begin{pmatrix} 0 & 0 \\ 0 & -r^{-1} \end{pmatrix}\right] \begin{pmatrix} v^r \\ v^\varphi \end{pmatrix}. \end{aligned}$$

We now look at equation (2.2), i.e. at $\dot v^\alpha = -(\dot\varphi \Gamma^\alpha{}_{\varphi\beta} + \dot r \Gamma^\alpha{}_{r\beta}) v^\beta$, and obtain

$$\begin{aligned} &\Gamma^r{}_{rr} = 0\ , \qquad \Gamma^r{}_{r\varphi} = \Gamma^r{}_{\varphi r} = 0\ , \qquad \Gamma^r{}_{\varphi\varphi} = -r\ , \\ &\Gamma^\varphi{}_{rr} = 0\ , \qquad \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = r^{-1}\ , \qquad \Gamma^\varphi{}_{\varphi\varphi} = 0\ . \end{aligned}$$

Other solution: start from $\Gamma^i{}_{jk} = 0$ in cartesian coordinates and transform according to equation (2.5):

$$\bar\Gamma^a{}_{bc} = \frac{\partial \bar x^a}{\partial x^d} \frac{\partial^2 x^d}{\partial \bar x^c \partial \bar x^b}\ ,$$

which can be calculated as matrix multiplication between

$$\left(D\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}\right)^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r^{-1}\sin\varphi & r^{-1}\cos\varphi \end{pmatrix},$$

and the matrices

$$\left(\frac{\partial^2 x^i}{\partial \bar x^j \partial r}\right) = \begin{pmatrix} 0 & -\sin\varphi \\ 0 & \cos\varphi \end{pmatrix} \qquad \left(\frac{\partial^2 x^i}{\partial \bar x^j \partial \varphi}\right) = \begin{pmatrix} -\sin\varphi & -r\cos\varphi \\ \cos\varphi & -r\sin\varphi \end{pmatrix},$$

producing the result

$$\begin{pmatrix} \Gamma^r{}_{rr} & \Gamma^r{}_{r\varphi} \\ \Gamma^\varphi{}_{rr} & \Gamma^\varphi{}_{r\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r^{-1}\sin\varphi & r^{-1}\cos\varphi \end{pmatrix} \begin{pmatrix} 0 & -\sin\varphi \\ 0 & \cos\varphi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & r^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^r{}_{\varphi r} & \Gamma^r{}_{\varphi\varphi} \\ \Gamma^\varphi{}_{\varphi r} & \Gamma^\varphi{}_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r^{-1}\sin\varphi & r^{-1}\cos\varphi \end{pmatrix} \begin{pmatrix} -\sin\varphi & -r\cos\varphi \\ \cos\varphi & -r\sin\varphi \end{pmatrix} = \begin{pmatrix} 0 & -r \\ r^{-1} & 0 \end{pmatrix}.$$

2. Affine connections

Let ∇ be an affine connection, and $\nabla_X Y - \tilde{\nabla}_X Y = B(X,Y)$. The first claim is proven through (i-iii), which relate to the definition of an affine connection (pg. 17-18):

i) In view of the linearity of $\nabla_x y$ w.r.t. X, Y, that of the vector fields $\tilde{\nabla}_X Y$, $B(X,Y)$ w.r.t. X, Y are equivalent.

ii) It follows from

$$\begin{aligned} \nabla_{fX} Y - \tilde{\nabla}_{fX} Y &= B(fX,Y) \,, \\ f\nabla_X Y - f\tilde{\nabla}_X Y &= fB(X,Y) \end{aligned} \tag{2}$$

and $\nabla_{fX} Y = f\nabla_X Y$, that the f-linearity w.r.t. X of $\tilde{\nabla}_X Y$ and of $B(X,Y)$ are equivalent.

iii) It follows from

$$\nabla_X(fY) - \tilde{\nabla}_X(fY) = B(X,fY) \,,$$

equation (2) and the product rule $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, that the product rule for $\tilde{\nabla}$ is equivalent to the f-linearity of $B(X,Y)$ w.r.t. Y.

For the Christoffel symbols $\Gamma^i{}_{lj} = \langle e^i, \nabla_{e_l} e_j\rangle$ it follows $\Gamma^i{}_{lj} - \tilde{\Gamma}^i_{lj} = B^i{}_{lj}$: differences of Christoffel symbols transform as tensors, see also (2.5).

Application: For $\nabla^{(\alpha)} := (1-\alpha)\nabla + \alpha\tilde{\nabla}$ we have $\nabla_X Y - \nabla^{(\alpha)}_X Y = \alpha B(X,Y)$, i.e. $\nabla^{(\alpha)}$ is an affine connection. In particular it is possible to interpolate between two affine connections ∇, $\tilde{\nabla}$ (with $0 \le \alpha \le 1$).

3. Alternate view on parallel transport

i) Let $J : \bar{U} \to \mathbb{R}^n \,,\ \ p \mapsto \bar{x}$ be the coordinate map on $\bar{U}$, hence $K \circ J^{-1} : \bar{x} \mapsto x$ the given transition function. We want to compute the transition function

$$\tilde{K} \circ \tilde{J}^{-1} : (\bar{x}, \underline{\bar{X}}) \longmapsto (x, \underline{X}) \,, \tag{3}$$

where $\tilde{J}$ is defined by the corresponding eq. (1). Since $X \in T_p(M)$ is a vector, we have

$$X^i = \frac{\partial x^i}{\partial \bar{x}^j} \bar{X}^j$$

and the matrix of partial derivatives of (3) is

$$D\tilde{t}(\bar{x}, \underline{\bar{X}}) = \left(\begin{array}{c|c} \frac{\partial x}{\partial \bar{x}} & \frac{\partial x}{\partial \underline{\bar{X}}} \\ \hline \frac{\partial \underline{X}}{\partial \bar{x}} & \frac{\partial \underline{X}}{\partial \underline{\bar{X}}} \end{array}\right) = \left(\begin{array}{c|c} \frac{\partial x^i}{\partial \bar{x}^j} & 0 \\ \hline \left(\frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k} \bar{X}^k\right) & \frac{\partial x^i}{\partial \bar{x}^j} \end{array}\right) . \tag{4}$$

ii) The lift condition $\pi(X(t)) = \gamma(t)$ is equivalent to $X(t) \in T_{\gamma(t)}$.
Let the curve $\gamma(t)$ have coordinates $x(t)$ under K, hence $\dot{\gamma}(t) \in T_{\gamma(t)}(M)$ has $\dot{x}(t)$ (under K_*). The curve $X(t) \in TM$ has coordinates $(x(t), \underline{X}(t))$ under $\tilde{K}$, hence its tangent vector $\dot{X}(t) \in T_{X(t)}(TM)$ has $(\dot{x}(t), \underline{\dot{X}}(t))$ (under $\tilde{K}_*$). The condition $\dot{X}(t) = \sigma_{X(t)}(\dot{\gamma}(t))$ thus states

$$(\dot{x}(t), \underline{\dot{X}}(t)) = \big(\dot{x}(t), -\Gamma(\dot{x}(t), \underline{X}(t))\big) \,,$$

i.e.

$$\dot{X}^i(t) = -\Gamma^i{}_{lk}(x(t))\dot{x}^l(t)X^k(t)\,,$$

which is just eq. (2.3) characterizing a parallel transported vector.

iii) Let $\Upsilon \in T_X(TM)$ have coordinates $(\underline{V}, \underline{W})$ under $\tilde{K}_*$. They transform by the matrix (4):

$$W^p = \frac{\partial^2 x^p}{\partial \bar{x}^l \partial \bar{x}^k}\bar{X}^k\bar{V}^l + \frac{\partial x^p}{\partial \bar{x}^i}\bar{W}^i\,.$$

Suppose $\Upsilon = \sigma_X(Y)$, i.e., $(\underline{V}, \underline{W}) = (\underline{Y}, -\Gamma(\underline{Y}, \underline{X}))$ and similarly in the barred coordinates. Then

$$-\Gamma^p{}_{rq}Y^rX^q = \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^l}\bar{X}^k\bar{Y}^l - \frac{\partial x^p}{\partial \bar{x}^i}\bar{\Gamma}^i{}_{lk}\bar{Y}^l\bar{X}^k$$

or, by solving w.r.t the last term,

$$\frac{\partial x^p}{\partial \bar{x}^i}\bar{\Gamma}^i{}_{lk} = \Gamma^p{}_{rq}\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^l}$$

times $\bar{Y}^l\bar{X}^k$, where we used $X^q = (\partial x^q/\partial \bar{x}^k)\bar{X}^k$. This is the same as

$$\bar{\Gamma}^i{}_{lk} = \frac{\partial \bar{x}^i}{\partial x^p}(\Gamma^p{}_{rq}\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^l})$$

which is (2.5).