General relativity, solution sheet 11.

HS 08

1. Time delay in the Schwarzschild metric

i) For null geodesics, the Lagrangian vanishes,

$$\frac{1}{1-2m/r}\mathcal{E}^2 - \frac{1}{1-2m/r}\dot{r}^2 - \frac{l^2}{r^2} = 0\,.$$

Hence, the effective potential is $V(r) = (1-2m/r)(l^2/r^2)$ and, since $\dot{r} = 0$ at $r = r_0$,

$$\frac{l^2}{\mathcal{E}^2} = \frac{r_0^2}{1-2m/r_0}\,.$$

Moreover, $\dot{r} = \dot{t}\,dr/dt = \mathcal{E}(1-2m/r)^{-1}\,dr/dt$, so that the radial equation is

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2m}{r}\right)^2 - \left(1-\frac{2m}{r}\right)^3 \frac{l^2}{\mathcal{E}^2\,r^2}\,.$$

Thus,

$$\left(\frac{dt}{dr}\right) = \frac{1}{1-\frac{2m}{r}}\left[1-\frac{1-2m/r}{1-2m/r_0}\left(\frac{r_0}{r}\right)^2\right]^{-1/2}\,.$$

ii) To first order in m/r,

$$1-\frac{1-2m/r}{1-2m/r_0}\left(\frac{r_0}{r}\right)^2 = \left[1-\left(\frac{r_0}{r}\right)^2\right]\left(1-\frac{2m}{r}\frac{r_0}{r+r_0}+\dots\right)\,,$$

so that

$$t = \int_{r_0}^{r} dr\left[1-\left(\frac{r_0}{r}\right)^2\right]^{-1/2}\left(1+\frac{2m}{r}+\frac{m}{r}\frac{r_0}{r+r_0}+\dots\right)$$

and the Shapiro time delay is

$$\Delta t(r) = 2m\log\left(\frac{r+\sqrt{r^2-r_0^2}}{r_0}\right) + m\left(\frac{r-r_0}{r+r_0}\right)^{1/2} + \dots\,.$$

iii) Finally, to lowest order in $r_0/r_{1,2}$,

$$\Delta t \approx 2\left(2m\log\frac{2r_1}{r_0} + m + 2m\log\frac{2r_2}{r_0} + m\right) = 4m\left(1+\log\frac{4r_1r_2}{r_0^2}\right)\,.$$

2. Radial free fall

The particle being initially at rest at $r(0) = R$, it has $\mathcal{E}^2 = 1-2m/R$ and $l = 0$. In this case, the radial equation

$$\dot{r}^2 - \frac{2m}{r} = -\frac{2m}{R}$$

is of the same form as the Friedmann equation for $k = 1$ but with different boundary conditions. In fact in terms of $s = \sqrt{2m/R}\,\tau$ the equation is

$$\left(\frac{dr}{ds}\right)^2 - \frac{R}{r} = -1$$

with the particular (expanding) cycloid solution (6.25)

$$\begin{aligned} r(\eta) &= \frac{R}{2}\,(1 - \cos\eta)\,, \\ s(\eta) &= \frac{R}{2}\,(\eta - \sin\eta)\,. \end{aligned}$$

The endpoints $\eta = 0, \pi$ correspond to $r = 0, R$ and $s = 0, (R/2)\pi$. However we seek the infalling solution, preferably parametrized forward in time. We thus replace $s \rightsquigarrow (R/2)\pi - s$, $\eta \rightsquigarrow \pi - \eta$ and obtain

$$\begin{aligned} r(\eta) &= \frac{R}{2}(1 + \cos\eta)\,, \\ \tau(\eta) &= \left(\frac{R^3}{8m}\right)^{1/2} (\eta + \sin\eta)\,, \end{aligned}$$

where $0 \leq \eta \leq \pi$. Note that the total proper time to reach $r = 0$ is finite, $\tau(\pi) = (\pi R/2)\sqrt{R/2m}$.