# General relativity, solution sheet 4.

*HS 08*

## 1. Euclidean metric in polar coordinates

The coordinate transformation is $x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$, and we have

$$dx^1 = \cos\varphi\, dr - r\sin\varphi\, d\varphi\,, \qquad dx^2 = \sin\varphi\, dr + r\cos\varphi\, d\varphi\,,$$

and $g = dr\otimes dr + r^2 d\varphi\otimes d\varphi$, i.e.

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}.$$

With ${\Gamma^i}_{kl} = {\Gamma^i}_{lk} = (1/2)g^{ij}(g_{lj,k} + g_{kj,l} - g_{lk,j})$ we can calculate the Christoffel symbols

$$\begin{aligned} {\Gamma^r}_{rr} &= \frac{1}{2}g_{rr,r} = 0\,, & {\Gamma^r}_{r\varphi} &= \frac{1}{2}g_{rr,\varphi} = 0\,, & {\Gamma^r}_{\varphi\varphi} &= \frac{1}{2}(2g_{\varphi r,\varphi} - g_{\varphi\varphi,r}) = -r\,, \\ {\Gamma^\varphi}_{rr} &= \frac{1}{2r^2}(2g_{r\varphi,\varphi} - g_{rr,\varphi}) = 0 & {\Gamma^\varphi}_{r\varphi} &= \frac{1}{2r^2}g_{\varphi\varphi,r} = r^{-1}\,, & {\Gamma^\varphi}_{\varphi\varphi} &= \frac{1}{2r^2}g_{\varphi\varphi,\varphi} = 0\,, \end{aligned}$$

in agreement with the solution of exercise 3.1.

## 2. Geodesics in the hyperbolic plane

a) The geodesics $(x(\tau), y(\tau))$ satisfy the variation principle $\delta\int_{(1)}^{(2)} d\tau (1/2) y^{-2}(\dot{x}^2 + \dot{y}^2) = 0$ (with fixed endpoints), i.e. the Hamiltonian variation principle with Lagrange function $L(y, \dot{x}, \dot{y}) = (1/2)y^{-2}(\dot{x}^2 + \dot{y}^2)$. With

$$\frac{\partial L}{\partial \dot{x}} = y^{-2}\dot{x}\,, \qquad \frac{\partial L}{\partial \dot{y}} = y^{-2}\dot{y}$$

the Euler-Lagrange equations are:

$$\ddot{x} - 2y^{-1}\dot{x}\dot{y} = 0\,, \qquad \ddot{y} + y^{-1}(\dot{x}^2 - \dot{y}^2) = 0\,. \tag{2}$$

b) As $L$ is indepentent of $\tau, x$, the conserved quantities are

$$\dot{x}(\partial L/\partial \dot{x}) + \dot{y}(\partial L/\partial \dot{y}) - L = L = y^{-2}(\dot{x}^2 + \dot{y}^2)\,, \qquad \frac{\partial L}{\partial \dot{x}} = y^{-2}\dot{x}\,. \tag{3}$$

c) The (Euclidean) curvature $\rho$ of a planar curve is the rate of change of the tangential vector: $|\rho| = |(d/ds)(x', y')|$, where $' = d/ds$ is the derivative w.r.t. the euclidean arc-length; a positive sign of $\rho$ corresponds to a left curve. As $s$ is the arc-length, the length of the tangent vector is $x'^2 + y'^2 = 1$, and $(x'', y'')$ is perpendicular to it, i.e. $x'x'' + y'y'' = 0$, and therefore parallel to the unit (left) normal vector $(-y', x')$. We have therefore

$$\rho = (x', y')\cdot(-y', x') = -y'x'' + x'y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\,,$$

where the last expression refers to a general curve parameter $\tau$ (use $\ddot{x} = (ds/d\tau)^2 x'' + (d^2 s/d\tau^2) x'$). Inserting (2) we find

$$\rho = -\frac{y^{-1}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = -\frac{y^{-2}\dot{x}}{y^{-1}(\dot{x}^2 + \dot{y}^2)^{1/2}} \, ,$$

what, according to (3) is conserved along the curve. Therefore the curve is a circle, and the tangent becomes vertical, $\dot{x} = 0$, as $y \to 0$, (see (3)): the center lies on the line $y = 0$.

## 3. An affine connection on Lie groups

Let $e_1, \ldots e_n$ be left-invariant basis fields with structure constants $C^i{}_{jk} = -C^i{}_{kj}$. We write equations corresponding to conditions (i,ii):

i) the condition is $\nabla_V V = 0$ for every left-invariant vector field $V$. Writing

$$V = V^i e_i = \lambda_{g*} V = (\lambda_{g*} V^i)(\lambda_{g*} e_i) = (\lambda_{g*} V^i) e_i$$

we conclude $V^i(g) = V^i(e)$ for the functions $V^j$. We have then

$$0 = \nabla_V V = V^i (V^j \nabla_{e_i} e_j + e_i(V^j) e_j) = V^i V^j \nabla_{e_i} e_j \, ,$$

where we may choose the $V^i$ freely, i.e.

$$\nabla_{e_i} e_j + \nabla_{e_j} e_i = 0 \, . \tag{4}$$

ii) The vanishing of the torsion

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

is equivalent to

$$\nabla_{e_j} e_k - \nabla_{e_k} e_j = C^i{}_{jk} e_i \, . \tag{5}$$

On the other hand definition of $\nabla$ through (1) is equivalent to

$$\begin{aligned} \nabla_{e_j} e_k + \nabla_{e_k} e_j &= \frac{1}{2}(C^i{}_{jk} + C^i{}_{kj}) e_i = 0 \\ \nabla_{e_j} e_k - \nabla_{e_k} e_j &= \frac{1}{2}(C^i{}_{jk} - C^i{}_{kj}) e_i = C^i{}_{jk} e_i \, , \end{aligned}$$

showing that the so defined $\nabla$ satisfies (i,ii). The derivation of (4),(5) shows that they are not only necessary but also sufficient conditions for (i,ii), proving therefore (a,b).