Exercise Sheet V

Hand in by 05.11.2008

Problem 1 [*Central extension in the Virasoro Algebra.*]: The goal of this exercise is to show that the Virasoro algebra is the only non-trivial central extension of the Witt algebra. Start from the general extension

$$[L_m, L_n] = (m-n)L_{m+n} + c_{m,n},$$

where the $c_{m,n} \in \mathbb{C}$ are in the center of the algebra, i.e. they commute with all other elements of the algebra.

(a) Use the antisymmetry of the Lie product and the Jacobi identity to prove that $c_{m,n}$ is antisymmetric in m and n and satisfies the equation

$$(n-k)c_{m,n+k} + (k-m)c_{n,m+k} + (m-n)c_{k,m+n} = 0. \tag{1}$$

(b) Prove that the terms $c_{n,0}$, $c_{0,n}$ and $c_{1,-1}$ can be set to zero by the trivial redefinition of the algebra

$$\tilde{L}_n = L_n + \frac{c_{n,0}}{n} \qquad n \neq 0$$
$$\tilde{L}_0 = L_0 + \frac{1}{2}c_{1,-1}.$$

(c) Setting $k = 0$ in equation (1) prove that $c_{m,n} = 0$ for $m + n \neq 0$.

(d) Use (1) to obtain a recursion relation for the coefficients $c_{m,-m}$, and solve it.

Problem 2 [*Transformation properties of the transversal string fields.*]: Show that the operator

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu\right)$$

acts on the transversal string fields X^I like its corresponding Lorentz transformation

$$[M^{ij}, X^I] = i(\delta^{jI}X^i - \delta^{iI}X^j).$$