Sheet 9

Due date: 16 May 2014

Exercise 1 [*Non-commuting observables*]: Let us consider the quantum mechanical system described by the Hamiltonian corresponding to the Hermitian matrix

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

acting on a two-dimensional Hilbert space. Another observable is given by the Hermitian matrix

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

(i) Compute the possible outcomes of measurements of the observable R.

(ii) Suppose that we measure the observable R at time $t = 0$, and find the value $R = +1$. After that the system evolves freely, and at time T we measure the observable R again. What is the probability to find the same value $R = +1$ as in the first measurement?

Exercise 2 [*Uncertainty relation*]: A particle of mass m moves in one dimension subject to the potential $\frac{1}{2}kx^2$ ($k > 0$). Express the expectation value of the energy E in terms of $\langle x\rangle$, $\langle p\rangle$, Δx and Δp. Hence, using the uncertainty relation $\Delta x \cdot \Delta p \geq \hbar/2$, show that

$$\langle E\rangle \geq \tfrac{1}{2}\hbar \left(\frac{k}{m}\right)^{1/2} .$$

This implies that there is a nonzero lower bound for the energy.

Exercise 3 [*Degeneracies*]: Let D^k_n be the number of ways of writing n as the sum of k non-negative integers $n_1, n_2, \ldots, n_k$. Show that the generating function $F^k(s)$ can be written as

$$F^k(s) := \sum_{n=0}^{\infty} D^k_n s^n = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} s^{n_1+\cdots+n_k} .$$

Deduce that $F^k(s) = (1 - s)^{-k}$, and hence or otherwise show that

$$D^k_n = \binom{n + k - 1}{n} .$$

Deduce the formulae for the degeneracy of energy eigenstates of isotropic harmonic oscillators in two and three dimensions.