Sheet 4

Due date: 21 March 2014

Exercise 1 [*Angular momentum of the electromagnetic field*]: In the absence of any charges and currents the angular momentum carried by the electromagnetic field is given by

$$\mathbf{L} = \frac{1}{4\pi k\, c} \int d^3\mathbf{x} \left[\mathbf{x} \wedge \Big(\mathbf{E}(\mathbf{x},t) \wedge \mathbf{B}(\mathbf{x},t) \Big) \right] .$$

(i) Under the assumption that the support of the $\mathbf{E}$ and $\mathbf{B}$ fields is compact, show that

$$\mathbf{L} = \frac{1}{4\pi k\, c} \int d^3\mathbf{x} \left[\mathbf{E}(\mathbf{x},t) \wedge \mathbf{A}(\mathbf{x},t) + \sum_{j=1}^{3} E_j(\mathbf{x},t)\, (\mathbf{x} \wedge \nabla) A_j(\mathbf{x},t) \right] ,$$

where $\mathbf{A}$ is the vector potential. The first term in this formula can be interpreted as the contribution of the photon spin to the total angular momentum, while the second term is the orbital angular momentum.

Hint: Use the identities $\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$, and write the magnetic field as $\mathbf{B} = \nabla \wedge \mathbf{A}$.

(ii) Let us consider the case where, in Coulomb gauge, the vector potential takes the form of a plane wave

$$\mathbf{A}(z,t) = \mathrm{Re}\left(A_+ \epsilon_+ e^{i(kz-\omega t)} + A_- \epsilon_- e^{i(kz-\omega t)} \right) ,$$

where $\epsilon_\pm = \frac{1}{\sqrt{2}}(\mathbf{e_x} \pm i\mathbf{e_y})$ and $A_\pm$ are constant parameters. Verify that this potential satisfies indeed the Coulomb gauge condition $\mathrm{div}\mathbf{A} = 0$, and determine the corresponding electric and magnetic fields. Show that they solve the vacuum Maxwell equations.

(iii) For the case of (ii), show that the time average of the first term of the angular momentum (the 'spin' term) equals

$$\mathbf{L}_{\mathrm{spin}} = \frac{1}{4\pi kc} \frac{|\mathbf{k}|}{2} \mathbf{e}_z \left(|A_+|^2 - |A_-|^2 \right) .$$

Interpret this result.

Exercise 2 [*Far zone approximation*]: In Lorentz gauge Maxwell's equations take the form

$$\Box\, \Phi = 4\pi\, k\, \rho\, , \qquad \Box\, \mathbf{A} = \frac{4\pi\, k}{c} \mathbf{j}\, ,$$

where Φ and $\mathbf{A}$ are the scalar and vector potentials, respectively. For the case where the current distribution is given by

$$\mathbf{j}(x,y,z,t) = \begin{cases} \mathrm{Re}\Big((d - |z|) e^{i\omega t} \Big) \delta(x)\delta(y)\, \mathbf{e}_z & |z| \le d \\ 0 & \text{otherwise}\, , \end{cases}$$

compute the magnetic field in the 'wave zone', ie., for

$$r \equiv \sqrt{x^2 + y^2 + z^2} \gg d \ , \qquad \frac{\omega}{c} \gg \frac{1}{r} \ .$$

Here ω is the (typical) inverse time during which the current distribution varies.