Exercise 1. ***Polarization of a neutral Fermi liquid***

Consider a system of neutral spin-1/2 particles each carrying a magnetic moment $\boldsymbol{\mu} = \frac{\mu}{2}\boldsymbol{\sigma}$. An electric field $\boldsymbol{E}$ couples to the particles by the relativistic spin-orbit interaction

$$H_{SO} = \frac{\mu}{2}\left(\frac{\boldsymbol{v}}{c} \times \boldsymbol{E}\right) \cdot \boldsymbol{\sigma}, \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is the vector of Pauli spin matrices. In the following we want to calculate the linear response function χ for the uniform polarization

$$\boldsymbol{P} = \chi \boldsymbol{E}. \tag{2}$$

In the presence of the spin-orbit interaction we have to consider a more general situation of a distribution of quasiparticles with variable spin quantization axis. In such a case we must treat the quasiparticle distribution function and the energy as a 2×2 matrix, $(\hat{n}_{\boldsymbol{p}})_{\alpha\beta}$ and $(\hat{\epsilon}_{\boldsymbol{p}})_{\alpha\beta}$, respectively. Furthermore, we require f to be a scalar under spin rotations. In this case f must be of the form

$$\hat{f}_{\alpha\beta,\alpha'\beta'}(\boldsymbol{p},\boldsymbol{p}') = f^s(\boldsymbol{p},\boldsymbol{p}')\delta_{\alpha\beta}\delta_{\alpha'\beta'} + f^a(\boldsymbol{p},\boldsymbol{p}')\boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{\sigma}_{\alpha'\beta'}. \tag{3}$$

(a) Expand $\hat{n}_{\boldsymbol{p}}$, $\hat{\epsilon}_{\boldsymbol{p}}$, and $\hat{f}_{\boldsymbol{\sigma\sigma}'}(\boldsymbol{p},\boldsymbol{p}')$ in terms of the unit matrix $\sigma^0 = \mathbf{1}$ and the Pauli spin matrices $\sigma^1 = \sigma^x$, $\sigma^2 = \sigma^y$, $\sigma^3 = \sigma^z$ and write down Landau's energy functional E.

(b) Assume that the electric field is directed along the z direction, $\boldsymbol{E} = E_z\hat{z}$. Show that the polarization of such a system is given by

$$P_z = \frac{\partial E}{\partial E_z} = \frac{\mu}{m^* c}\sum_{\boldsymbol{p}}\left(p_y \delta n^1_{\boldsymbol{p}} - p_x \delta n^2_{\boldsymbol{p}}\right). \tag{4}$$

Here, $\delta n^i_{\boldsymbol{p}} = \frac{1}{2}\mathrm{tr}\left[\delta\hat{n}_{\boldsymbol{p}}\,\sigma^i\right]$ and $\delta\hat{n}_{\boldsymbol{p}}$ is the deviation from the equilibrium ($E_z = 0$) distribution function.

(c) The application of the electric field changes the quasiparticle energy in linear response according to

$$\delta\tilde{\epsilon}^i_{\boldsymbol{p}} = \delta\epsilon^i_{\boldsymbol{p}} + \frac{2}{V}\sum_{\boldsymbol{p}'} f^{ii}(\boldsymbol{p},\boldsymbol{p}')\delta n^i_{\boldsymbol{p}'} \quad \text{with} \quad \delta n^i_{\boldsymbol{p}} = \frac{\partial n^0_{\boldsymbol{p}}}{\partial \epsilon_{\boldsymbol{p}}}\delta\tilde{\epsilon}^i_{\boldsymbol{p}} = -\delta_D(\epsilon^0_{\boldsymbol{p}} - \epsilon_F)\delta\tilde{\epsilon}^i_{\boldsymbol{p}}, \tag{5}$$

where δ_D denotes the Dirac delta function.

Use the ansatz $\delta\tilde{\epsilon}^i_{\boldsymbol{p}} = \alpha\delta\epsilon^i_{\boldsymbol{p}}$ and show that $\alpha = 1/(1 + F^a_1/3)$.

(d) Compute χ according to Eq. (2).