Exercise 1. ***Lindhard function***

In the lecture it was shown how to derive the dynamical linear response function $\chi_0(\boldsymbol{q}, \omega)$ which is also known as the Lindhard function:

$$\chi_0(\boldsymbol{q}, \omega) = \frac{1}{\Omega} \sum_{\boldsymbol{k}} \frac{n_{0,\boldsymbol{k}+\boldsymbol{q}} - n_{0,\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}+\boldsymbol{q}} - \epsilon_{\boldsymbol{k}} - \hbar\omega - i\hbar\eta}. \tag{1}$$

Calculate the static Lindhard function $\chi_0(\boldsymbol{q}, \omega = 0)$ of free electrons for the (a) 1- and (b) 3-dimensional cases at $T = 0$. Show that $\chi_0(\boldsymbol{q}, \omega = 0)$ has a vanishing imaginary part.

Hint. For the calculation of the real part of $\chi_0(\boldsymbol{q}, \omega)$ *you may use the equation* $\lim_{\eta \to 0}(z - i\eta)^{-1} = \mathcal{P}(1/z) + i\pi\delta(z)$. *Note that in 3 dimensions we can choose* $\boldsymbol{q} = q\boldsymbol{e}_z$ *to point in the z-direction due to the isotropy of a system of free electrons. Also, changing to cylindrical coordinates in order to calculate the integral turns out to be helpful.*

Exercise 2. ***Zero-sound excitations***

The dispersion relation of the plasmon excitation is finite for all $\boldsymbol{q}$'s. This appearance of a finite excitation energy is a consequence of the long range character of the Coulomb potential $V_{\text{Coulomb}}(\boldsymbol{r}) = e^2/|\boldsymbol{r}|$. A system consisting of fermions with a purely local potential

$$V_{\text{local}}(\boldsymbol{r}) = U \cdot \delta(\boldsymbol{r}) \tag{2}$$

shows a different behavior at $\boldsymbol{q} = 0$. In this exercise we will follow sections (3.2.1) to (3.2.3) of the lecture notes in order to discuss the plasmon excitation in the context of a local interaction.

(a) As a warm-up, derive the relation between the particle distribution $\delta n(\boldsymbol{r}, t)$ and its induced potential $V_{\text{ind}}(\boldsymbol{r}, t)$ in the $(\boldsymbol{k}, \omega)$-space.

(b) Find the imaginary part of the response function $\chi(\boldsymbol{q}, \omega)$ for small $\boldsymbol{q}$'s. What is the dispersion relation to lowest order in $\boldsymbol{q}$?

(c) The upper boundary line of the particle-hole continuum is given by

$$\hbar\omega_{q,\max} = \frac{\hbar^2}{2m}\left(q^2 + 2k_{\text{F}}q\right) = \frac{\hbar^2 q^2}{2m} + \hbar v_F q, \tag{3}$$

where v_F is the Fermi velocity and $q = |\boldsymbol{q}|$. What is the condition on U in order to obtain stable plasmon excitations (quasi-particles)?

This collective mode was first predicted by Landau in 1957 in the framework of his theory of Fermi liquids. In 1966 zero sound was experimentally observed in He^3 by Abel, Anderson and Wheatley. References :

- L. D. Landau, JETP 32, 59 (1957), Soviet Phys. JETP 5, 101 (1957).
- W. R. Abel, A. C. Anderson, and J. C. Wheatley, Propagation of Zero Sound in Liquid He^3 at Low Temperatures, Phys. Rev. Lett. 17, 74-78 (1966).
- L. P. Pitaevskii, Zero Sound in Liquid He^3, Sov. Phys. Usp. , **10, 100 (1967).**

Exercise 3. ***Particle–hole spectrum of a one-dimensional metal***

Sketch the particle–hole excitation spectrum for a 1-dimensional metal and compare it to the already known case of a 3-dimensional system. Assume a free-electron dispersion

$$\epsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m}. \tag{4}$$