EXERCISE 1

The Lagrangian of a scalar field is of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_n)^2 + V(\phi_n)$$

where also $V(\phi_n)$ is symmetric under S. This symmetry acts on the fields as

$$\delta\phi_m = i\, t_{mn}\, \phi_n$$

In order to show that $[Q, \phi_n(x)] = -t_{nm}\phi_m(x)$, let us first find the Noether current associated to S and the corresponding charge Q.

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_m)}\,\delta\phi_m = \partial^\mu\phi_m\; i(t_{mn})\phi_n$$

hence $Q(t) = \int d\bar{x}\;\partial^0\phi_m\; i(t_{mn})\phi_n$

We notice that the conjugate momentum of ϕ_n is

$$\pi_m = \frac{\delta\mathcal{L}}{\delta\dot{\phi}_m} = \partial^0\phi_m \quad , \text{ with } \quad [\pi^m(\bar{x}), \phi^n(\bar{y})] = i\delta(\bar{x}-\bar{y})\,\delta^{mn}$$

$$[\phi_m(\bar{x}), \phi_n(\bar{y})] = 0$$

so that $\quad Q = \int d\bar{x}\;\pi_m\; i(t_{mn})\phi_n$.

Hence we have

$$[Q, \phi_n(x)] = \int d\bar{y}\;[\pi_m(\bar{y}), \phi_n(\bar{x})]\; i\,t_{m\varsigma}\,\phi_\varsigma(\bar{y})$$

$$= -t_{n\varsigma}\,\phi_\varsigma(x)$$

EXERCISE 2

The Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{n} (\partial_\mu \phi_n)^2 - \frac{\mu^2}{2} \sum_{n} \phi_n^2 - \frac{\lambda}{4} \left(\sum_{n} \phi_n\right)^4$$

of N real scalar fields ϕ_n, $n = 1, N$ is symmetric under the rotation $O(N)$ group

$$\phi_n \to \phi'_n = R_{nm} \phi_m \quad , \quad R^2 = 1$$

The field vacuum expectation values can be computed by minimizing the effective potential

$$\left.\frac{\partial V_{eff}}{\partial \phi_e}\right|_{\phi_e = \langle \phi_e \rangle} = 0$$

Ignoring loop effects, V_{eff} is equal to the classical potential

$$V_{eff}(\phi) = \frac{\mu^2}{2} \sum_{n} \phi_n^2 + \frac{\lambda}{4} \left(\sum_{n} \phi_n\right)^4$$

whose extrema are solutions of

$$\langle \phi_e \rangle \left(\mu^2 + \lambda \sum_{n} \langle \phi_n \rangle^2 \right) = 0$$

For $\mu^2 > 0$ and $\lambda > 0$, the minimum is at $\langle \phi_e \rangle = 0$, ~~while~~ which is invariant under $O(N)$, while for $\mu^2 < 0$, $\lambda > 0$ the minima satisfy

$$\sum_{n} \langle \phi_n \rangle^2 = -\frac{\mu^2}{\lambda}$$

they are degenerate and not invariant under $O(N)$: S.S.B has occurred. Only one minimum is chosen spontaneously, but it could be any of them.

The mass matrix

$$M_{nm} = \frac{\partial^2 V_{eff}}{\partial\phi_n \partial\phi_m}\Bigg|_{\phi_{m,n} = \langle\phi_e\rangle} = \mu^2 \delta_{mn} + \lambda \delta_{nm} \sum_e \langle\phi_e\rangle^2 + 2\lambda \langle\phi_n\rangle\langle\phi_m\rangle$$

$$= 2\lambda \langle\phi_n\rangle\langle\phi_m\rangle$$

since the first term vanishes because of the condition $\sum_n \langle\phi_n\rangle^2 = -\frac{\mu^2}{\lambda}$.

Therefore, we have to diagonalize the matrix

$$M_{nm} = 2\lambda \langle\phi_n\rangle\langle\phi_m\rangle = \begin{pmatrix} \langle\phi_1\rangle^2 & \cdots & \langle\phi_n\rangle\langle\phi_1\rangle \\ \vdots & \ddots & \\ \langle\phi_1\rangle\langle\phi_m\rangle & \cdots & \langle\phi_n\rangle^2 \end{pmatrix}$$

Then

$$\det(M_{nm} - \mu^2 \delta_{mn}) = 0$$

$$\Rightarrow (\mu^2)^{N-1}\left(\mu^2 - 2\lambda \sum_i \langle\phi_i\rangle^2\right) = 0$$

There is only one massive particle with mass

$$\mu^2 = 2\lambda \sum_i \langle\phi_i\rangle^2 = 2|\mu|^2$$

and $(N-1)$ Goldstone boson particle, which are massless. Why $(N-1)$? This is because $O(N)$ has $\frac{1}{2}N(N-1)$ generators, while after the symmetry is broken, the remaining symmetry is $O(N-1)$, since one direction has been picked up in the $\langle\phi_i\rangle$ space. ~~The linear combinations of fields which are orthogonal to this direction~~ Therefore, the number of broken generators is $\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1$.

The generic scalar 2-loop integral in d-dimension with n massless scalar propogators $\frac{1}{A_i}$ raised to some powers ν_i is given by

$$J^d(\{\nu_i\}, \{Q^2_i\}) = \int \frac{d^d k_1}{i \pi^{d/2}} \int \frac{d^d k_2}{i \pi^{d/2}} \frac{1}{A_1^{\nu_1} \cdots A_n^{\nu_n}}$$

Using Feynman parameters to rewrite the product of propogators, we find

$$J^d(\{\nu_i\}, \{Q^2_i\}) = \frac{\Gamma(\nu_1 + \cdots + \nu_n)}{\Gamma(\nu_1) \cdots \Gamma(\nu_n)} \int \frac{d^d k_1}{i \pi^{d/2}} \int \frac{d^d k_2}{i \pi^{d/2}} \cdot$$

$$\cdot \int_0^1 dx_1 \ldots dx_n \frac{\delta(1 - \sum_{i=1}^{n} x_i) \; x_1^{\nu_1 - 1} \cdots x_n^{\nu_n - 1}}{\left[\sum_i x_i A_i\right]^{\nu_1 + \cdots \nu_n}}$$

where the most general form for the sum in the denominator is given by

$$\sum_i x_i A_i = a k_1^2 + b k_2^2 + 2c \, k_1 \cdot k_2 + 2d \cdot k_1 + 2e \cdot k_2 + f$$

where a, b, c, d^μ, e^μ, f are linear in the x_i.

If we now change variables according to

$$\begin{cases} k_1^\mu = Q_1^\mu - \frac{c \, Q_2^\mu}{a} + X^\mu \\ k_2^\mu = Q_2^\mu + Y^\mu \end{cases}$$

with $\quad X^\mu = \dfrac{c e^\mu - b d^\mu}{p} \qquad Y^\mu = \dfrac{c d^\mu - a e^\mu}{p}$

$$p = ab - c^2$$

we have

$$\sum_i x_i A_i = a \left(Q_1^\mu - \frac{c}{a} Q_2^\mu + X^\mu \right)^2 + b \left(Q_2^\mu + Y^\mu \right)^2$$

$$+ 2c \left(Q_1^\mu - \frac{c}{a} Q_2^\mu + X^\mu \right) \cdot \left(Q_2^\mu + Y^\mu \right)$$

$$+ 2d \cdot \left(Q_1^\mu - \frac{c}{a} Q_2^\mu + X^\mu \right) + 2e \cdot \left(Q_2^\mu + Y^\mu \right)$$

$$+ f$$

$$= a Q_1^2 + Q_2^2 \left(\frac{c^2}{a} + b - 2\frac{c^2}{a} \right) +$$

$$+ a X^2 + b Y^2 + 2c X \cdot Y + 2d \cdot X + 2e \cdot Y + f$$

$$= a Q_1^2 + Q_2^2 \frac{(ab - c^2)}{a} + \frac{1}{p} \left[- a e^2 - b d^2 + 2c(d \right.$$
$$\left. + f p \right]$$

$$= a Q_1^2 + \frac{p}{a} Q_2^2 + \frac{Q}{p}$$

where $Q = - a e^2 - b d^2 + 2 c e \cdot d + f p$

b) The integral has now the form

$$J^d(\{\nu_i\},\{Q_i^2\}) = \frac{\Gamma(N)}{\prod_i \Gamma(\nu_i)} \int_0^1 dx_1 \dots dx_m \; \delta(1-\sum_i x_i)\, x_1^{\nu_1-1}\dots x_m^{\nu_m}$$

$$\cdot \int \frac{d^d Q_1}{i\pi^{d/2}} \int \frac{d^d Q_2}{i\pi^{d/2}} \frac{1}{\left[a Q_1^2 + \frac{\rho}{a} Q_2^2 + \frac{Q}{\rho}\right]^N}$$

with $N = \sum_i \nu_i$

We can perform first the integration in Q_1 :

$$\int \frac{d^d Q_1}{i\pi^{d/2}} \frac{1}{a^N\left[Q_1^2 + \frac{\rho}{a^2} Q_2^2 + \frac{Q}{\rho a}\right]^N} =$$

$$= \int \frac{d^d Q_1}{i\pi^{d/2}} \frac{1}{a^N\left[Q_1^2 - \Delta\right]^N} = \frac{1}{a^N} (-1)^N \frac{\Gamma(N-d/2)}{\Gamma(N)} \Delta^{d/2-N}$$

and then the integral in Q_2 :

$$\int \frac{d^d Q_2}{i\pi^{d/2}} \frac{(-1)^N}{a^N} \frac{\Gamma(N-d/2)}{\Gamma(N)} \frac{1}{\left[\frac{\rho}{a^2} Q_2^2 + \frac{Q}{\rho a}\right]^{N-d/2}} =$$

$$= \frac{(-1)^{d/2}}{a^N} \frac{\Gamma(N-d/2)}{\Gamma(N)} \frac{1}{(\rho/a^2)^{N-d/2}} \int \frac{d^d Q_2}{i\pi^{d/2}} \frac{1}{\left[Q_2^2 + \underbrace{\frac{Q a}{\rho^2}}_{\Delta'}\right]^{N-d/2}}$$

$$\frac{(-1)^{d/2}}{a^N} \frac{\Gamma(N-d/2)}{\Gamma(N)} \left(\frac{a^2}{P}\right)^{N-d/2} (-1)^{N-d/2} \frac{\Gamma(N-d)}{\Gamma(N-d/2)} \frac{1}{(\Delta')^{N-d}}$$

$$= (-1)^d \frac{\Gamma(N-d)}{\Gamma(N)} P^{N-\frac{3}{2}d} Q^{d-N}$$

and hence we have :

$$J^d(\{\nu_i\}, \{Q^2_i\}) = (-1)^d \frac{\Gamma(N-d)}{\prod_i \Gamma(\nu_i)} \int_0^1 dx_1 \dots dx_n \, \delta(1-\sum_i x_i) \,. \, P^{N-\frac{3}{2}d} Q^{d-N}$$

c) In the case of 1-loop scalar integral, ~~with~~ we can deduce from the sum

$$\sum_i x_i A_i = a k_1^2 + b k_2^2 + 2c\, k_1 \cdot k_2 + 2\, d \cdot k_1 + 2\, e \cdot k_2 + f$$

that : first

1) taking $c \to 0$ in order to eliminate the common propagator of the two loops

2) after, setting $b = 0$ and $e^\mu = 0$ to eliminate the propagators of the second loop

we find $\frac{c}{P} \to 0$

$$\frac{b}{P} \to \frac{1}{a} \to \frac{1}{P} \Rightarrow P = a$$

$$x^\mu \longrightarrow -\frac{d^\mu}{\rho}$$

$$y^\mu \longrightarrow 0$$

$$\frac{Q}{\rho} \longrightarrow \frac{-d^2 + f\rho}{\rho} \quad \Rightarrow \quad Q = f - d^2$$

in this case $a = \sum_i x_i = 1$ because of the constraint due to the delta function for the Feynman parameters $\delta(1 - \sum_i x_i)$

d) In this case we have to integrate one time less than in the 2-loop case. Therefore, since

$$\sum_i x_i A_i = Q_1^2 + \frac{Q}{\rho} = Q_1^2 + Q$$

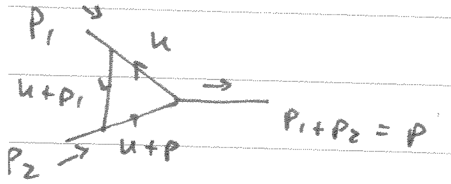
we find $J^d(\{v_i\}, \{Q_i^2\}) = (-1)^{d/2} \frac{\Gamma(N - d/2)}{\prod_i \Gamma(v_i)} \int_0^1 dx_1 \dots dx_M \cdot$

$$\cdot \, \delta(1 - \sum_i x_i) \, x_1^{v_1 - 1} \dots x_M^{v_M - 1} \, Q^{d/2 - N}$$

e) For the 1-loop scalar triangle with $p_1^2 = p_2^2 = 0$

$P = p_1 + p_2$

$s = P^2$



$$= \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 (k+p_1)^2 (k+p)^2}$$

$$\int \frac{d^d k}{i\pi^{d/2}} \int_0^1 dx_1\, dx_2\, dx_3 \frac{\delta(1-\sum_i x_i)}{\left[k^2 x_1 + (k+p_1)^2 x_2 + (k+p)^2 x_3\right]^3} \Gamma(3)$$

Look at the sum in the denominator:

$$\underbrace{k^2 (x_1+x_2+x_3)}_{a} + 2 k \cdot \underbrace{(p_1 x_2 + p x_3)}_{d^\mu} + \underbrace{x_2 p_1^2 + x_3 p^2}_{f}$$

$$d^\mu = \sum_i x_i q_i^\mu$$

$$f = \sum_i x_i q_i^2 \qquad \text{where } q_i^\mu = q_{i-1}^\mu + p_{i-1}^\mu$$

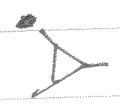
$$q_1^\mu = 0$$

$$= k^2 + 2 k \cdot d + f$$

Now let's shift the loop momentum:

$$\ell^\mu = k^\mu + d^\mu \quad \Rightarrow \quad \ell^2 = k^2 + 2 d \cdot k + d^2$$

The integral is then

 $$= \int_0^1 dx_1\, dx_2\, dx_3\, \delta(1-\sum_i x_i) \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{\left[\ell^2 - d^2 + f\right]^3} \Gamma(3)$$

$$= (-1)^{d/2} \Gamma(3-d/2) \int_0^1 dx_1\, dx_2\, dx_3\, \delta(1-\sum_i x_i)\, Q^{d/2-3}$$

where $Q = f - d^2 = s\, x_1 x_3$

$$Q = \left(\sum_{i=1}^{M} x_i q_i^2\right)\left(\sum_{j=1}^{M} x_j\right) - \left(\sum_{i=1}^{M} x_i q_i\right)\left(\sum_{j=1}^{M} x_j q_j\right)$$

$$= \sum_{i,j=1}^{M} x_i x_j \left(q_i^2 - q_i \cdot q_j\right)$$

$$= \frac{1}{2} \sum_{i,j=1}^{M} x_i x_j \left(q_i^2 - 2 q_i \cdot q_j + q_j^2\right)$$

$$= \frac{1}{2} \sum_{i,j=1}^{M} x_i x_j \left(q_i - q_j\right)^2$$

$$= \sum_{j=2}^{M} \sum_{i<j} x_i x_j s_{ij}$$

with $s_{ij} = (q_i - q_j)^2 = \left(\sum_{m=i}^{j-1} p_m^4\right)^2$