Sheet 12

Ex 1 (a) In $d = 4-2\epsilon$ dimensions the coupling parameter g_R acquires a mass dependence after renormalisation:

$$[g_R] = \epsilon .$$

$\Rightarrow$ Can write $g_R = \tilde{g}_R \mu^\epsilon$

where μ = renormalisation scale & now $[\tilde{g}_R] = 0$.

By abuse of notation one just writes: $g_R \mu^\epsilon$.

Then from the definition of g_R & Z_{1F} it follows that:

$$g_0 = Z_2^{-1} Z_A^{-\frac{1}{2}} Z_{1F}\, g_R \mu^\epsilon$$

(b) $Z_i = 1 + \delta Z_i \quad , \; i = A, 2, 1F \qquad$ where $\delta Z_i = O(g_R^2)$

then:
$$g_0 = (1+\delta Z_2)^{-1}(1+\delta Z_A)^{-\frac{1}{2}}(1+\delta Z_{1F})\, g_R\mu^\epsilon$$
$$= (1 - \delta Z_2 + O(g_R^4))(1 - \tfrac{1}{2}\delta Z_A + O(g_R^4)) \times (1+\delta Z_{1F})\, g_R \mu^\epsilon$$
$$\simeq \left(1 - \delta Z_2 - \tfrac{1}{2}\delta Z_A + \delta Z_{1F}\right) g_R \mu^\epsilon$$

From sheet 10:
$$\begin{cases} \delta Z_A = -\dfrac{g_R^2}{16\pi^2}\left(\tfrac{2}{3}n_f - \tfrac{5}{3}N_c\right)\left(\dfrac{1}{\epsilon} + \text{finite}\right) \\ \delta Z_2 = -\dfrac{g_R^2}{16\pi^2} C_F\left(\dfrac{1}{\epsilon} + \text{finite}\right) \end{cases}$$

From sheet 11:
$$\delta Z_{1F} = -\frac{g_R^2}{16\pi^2}(N_c + C_F)(4\pi)^\epsilon \Gamma(\epsilon)$$
$$\simeq -\frac{g_R^2}{16\pi^2}(N_c + C_F)\left(\frac{1}{\epsilon} + \text{finite}\right) \quad (\text{in } \xi = 1 \text{ gauge}).$$

$$\Rightarrow g_0 = \left(1 + \frac{g_R^2}{16\pi^2}\cancel{C_F}\cdot\frac{1}{\epsilon} + \frac{1}{2}\cdot\frac{g_R^2}{16\pi^2}\left(\tfrac{2}{3}n_f - \tfrac{5}{3}N_c\right)\cdot\frac{1}{\epsilon} - \frac{g_R^2}{16\pi^2}(N_c + \cancel{C_F})\frac{1}{\epsilon} + \text{finite}\right) g_R\mu^\epsilon$$
$$= \left[1 - \frac{g_R^2}{16\pi^2}\left(\tfrac{11}{6}N_c - \tfrac{1}{3}n_f\right)\left(\frac{1}{\epsilon} + \text{finite}\right)\right] g_R \mu^\epsilon$$

Exercise 2

(a) Using the result for g_0 in Ex 1 one has:

$$\mu \frac{\partial}{\partial \mu} g_0 \Big|_{g_0 = \text{const}} = 0 \overset{!}{=} \epsilon \mu^{\epsilon} \left[1 - \frac{g_R^2}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} n_f \right) \left(\frac{1}{\epsilon} + \text{finite} \right) \right] g_R$$

$$+ \mu^{\epsilon} \left[1 - \frac{3 g_R^2}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} n_f \right) \left(\frac{1}{\epsilon} + \text{finite} \right) \right] \beta$$

$$\Rightarrow \beta = - \epsilon g_R \left[1 - \frac{g_R^2}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} n_f \right) \left(\frac{1}{\epsilon} + \text{finite} \right) \right]$$
$$\times \left[1 - \frac{3 g_R^2}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} n_f \right) \left(\frac{1}{\epsilon} + \text{finite} \right) \right]^{-1}$$

$$= - \epsilon g_R \left[1 - \frac{g_R^2}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} n_f \right) \left(\frac{1}{\epsilon} + \text{finite} \right) \right]$$
$$\times \left[1 + \frac{3 g_R^2}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} n_f \right) \left(\frac{1}{\epsilon} + \text{finite} \right) + \ldots \right]$$
$$+ O(g_R^5)$$

Let $\epsilon \to 0$
$$= - \frac{2 \cdot g_R^3}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} n_f \right) + O(g_R^5)$$

$$= - \left(\frac{11}{3} N_c - \frac{2}{3} n_f \right) \frac{g_R^3}{16\pi^2} + O(g_R^5)$$

(b) Let $b_0 = \frac{11}{3} N_c - \frac{2}{3} n_f$

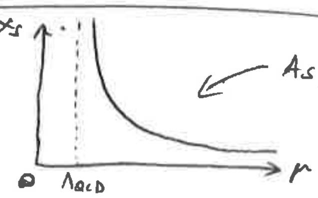
then: $\mu \frac{\partial g_R}{\partial \mu} = - \frac{g_R^3}{16\pi^2} b_0 \Rightarrow g_R^2 = \frac{1}{\frac{b_0}{8\pi^2} \ln \mu + C}$, where C = const of integration.

Using: $\lim_{\mu \to \Lambda_{QCD}} \frac{1}{g_R^2(\mu)} = 0 \Rightarrow C = - \frac{b_0}{8\pi^2} \ln \Lambda_{QCD}$.

$$\Rightarrow g_R^2 = \frac{1}{\frac{b_0}{8\pi^2} \ln \left(\frac{\mu}{\Lambda_{QCD}} \right)}$$

$$\Rightarrow \alpha_s = \frac{g_R^2}{4\pi} = \frac{4\pi}{\left(\frac{11}{3} N_c - \frac{2}{3} n_f \right) \ln \left(\frac{\mu^2}{\Lambda_{QCD}^2} \right)}$$

(c) In QCD $N_c = 3$, $n_f = 6$ $\Rightarrow \beta < 0 \Rightarrow$



Asymptotic freedom!!

Hence quarks behave like free particles at high energies!