QFT II - Solution 6 / Ex 1 & 2.

1. a) The equation to be solved is

$$\left[(+\partial^2+\mu^2)\, g_{\mu\nu} - \left(1-\tfrac{1}{\xi}\right)\partial_\mu\partial_\nu\right] \Delta^{\nu\lambda}(x) = \delta_\mu{}^\lambda\, \delta^{(4)}(x)$$

Fourier-transform (or rather inverse Fourier transform)

$$\Delta^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \tilde{\Delta}^{\mu\nu}(k) \quad ; \quad \delta^{(4)}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}$$

$\hookrightarrow$ The differential equation becomes an algebraic equation:

$$\boxed{\left[(k^2+\mu^2)\, g_{\mu\nu} - \left(1-\tfrac{1}{\xi}\right) k_\mu k_\nu\right] \tilde{\Delta}^{\nu\lambda}(k) = \delta_\mu{}^\lambda}$$

We insert the most general rank-2 Lorentz-tensor as an ansatz for $\tilde{\Delta}$

$$\tilde{\Delta}^{\mu\nu} = A\, g^{\mu\nu} + B\, k^\mu k^\nu$$

$$\Rightarrow \delta_\mu{}^\lambda = \left[(k^2+\mu^2)\, g_{\mu\nu} - \left(1-\tfrac{1}{\xi}\right) k_\mu k_\nu\right]\left[A g^{\nu\lambda} + B k^\nu k^\lambda\right] =$$

$$= A(k^2+\mu^2)\,\delta_\mu{}^\lambda + \underbrace{\left[B(k^2+\mu^2) - A\left(1-\tfrac{1}{\xi}\right) - Bk^2\left(1-\tfrac{1}{\xi}\right)\right]}_{=(*)} k_\mu k^\lambda$$

i.e. $\boxed{A(k^2+\mu^2) \overset{!}{=} 1}$ and $\boxed{(*) \overset{!}{=} 0}$

$\Downarrow$ $\quad\quad\quad\quad$ $\Downarrow$

$$\underline{\underline{A = \frac{1}{k^2+\mu^2}}} \qquad B = \left(1-\tfrac{1}{\xi}\right)\frac{1}{k^2+\mu^2}\cdot\frac{1}{\frac{k^2}{\xi}+\mu^2} = \underline{\underline{\frac{1}{k^2+\mu^2}\cdot\frac{\xi-1}{k^2+\xi\mu^2}}}$$

So the solution is:

$$\boxed{\Delta^{\mu\nu}(x) = \int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ikx}}{k^2+\mu^2}\left(g^{\mu\nu} + \frac{\xi-1}{k^2+\xi\mu^2}\, k^\mu k^\nu\right)}$$

b) We add the term $-\frac{1}{2}w^2(x)$ to the free Yang-Mills Lagrangian (kinetic term of the gauge field), with $w(x) = n_\mu A^{\mu,a}$

$$\rightarrow \mathcal{L}_{YM+GF} = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a}\right) - \frac{1}{2}\left(n_\mu A^{\mu,a}\right)\left(n_\nu A^{\nu,a}\right) =$$

$$\overset{\text{P.S1}}{=} -\frac{1}{2} A^{\mu,a}\left[-\partial^2 g_{\mu\nu} + \partial_\mu\partial_\nu\right]\delta^{ab} A^{\nu,b} - \frac{1}{2} A^{\mu,a}\, n_\mu n_\nu\, A^{\nu,a} = \ldots$$

$$= -\frac{1}{2} A^{\mu,a} \underbrace{\left[-\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu + n_\mu n_\nu\right] \delta^{ab}}_{\text{To be inverted}} A^{\nu,b}$$

i.e. we're looking for $\Delta^{\mu\nu,ab}(x)$ s.t.:

$$\boxed{\left[-\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu + n_\mu n_\nu\right] \delta^{ab} \Delta^{\nu\lambda,bc}_{(x)} = \delta_\mu^{\ \lambda} \delta^{ac} \delta^{(4)}(x)}$$

The colour part can only be another Kronecker-delta, i.e.

$\boxed{\Delta^{\mu\nu,ab} = \Delta^{\mu\nu} \delta^{ab}}$ Then, as before, we go to Fourier space and choose an Ansatz for $\tilde{\Delta}^{\mu\nu}$. Here, we can build more terms because of n_μ:

$$\tilde{\Delta}^{\mu\nu}(k) = A g^{\mu\nu} + B k^\mu k^\nu + C n^\mu n^\nu + D k^\mu n^\nu + E n^\mu k^\nu$$

⇒ The equation becomes:

$$\delta_\mu^{\ \lambda} = \delta_\mu^{\ \lambda}(Ak^2) + k_\mu k^\lambda\left(-A + \cancel{k^2 B} - \cancel{k^2 B} - (n\cdot k)E\right) + n_\mu n^\lambda\left(A + (n\cdot k)D + k^2 C\right)$$
$$+ k_\mu n^\lambda\left(k^2 D - (n\cdot k)C - k^2 D\right) + n_\mu k^\lambda\left(k^2 E + (n\cdot k)B\right)$$

So $\boxed{A = \frac{1}{k^2}}$ and all the other brackets need to be zero. This yields

$$E = -\frac{1}{k^2 (n\cdot k)}, \quad B = -\frac{1}{(n\cdot k)^2}, \quad C = 0, \quad D = E$$

And thus: $\boxed{\Delta^{ab}_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2}\left[g_{\mu\nu} - \frac{k^2}{(n\cdot k)^2} k_\mu k_\nu - \frac{(n_\mu k_\nu + n_\nu k_\mu)}{(n\cdot k)}\right]\delta^{ab}}$

Sheet 6

Ex3

$$Z \propto \int DA_\mu \, D\bar\psi \, D\psi \, D\bar\eta \, D\eta \; \exp\,(i \underbrace{S_{free}}_{free} + i \underbrace{S_{int}}_{interaction} + i \underbrace{S_{src}}_{source})$$

$$S = \int d^4x \, \Big(\underbrace{-\tfrac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}}_{\mathcal{L}_{YM}} + \underbrace{\left(-\tfrac{1}{2\xi}\right) (\partial^\mu A^a_\mu)^2}_{\mathcal{L}_{gauge-fixing}} + \underbrace{(\partial^\mu \bar\eta^a) D^{ab}_\mu \eta^b}_{\mathcal{L}_{ghost}} + \underbrace{\bar\psi^i (i\gamma^\mu D^{ij}_\mu - m\,\delta^{ij})\psi^j}_{\mathcal{L}_{fermion}}$$

$$+ J^{\mu,a}_A A^a_\mu + J_\psi \psi + J_{\bar\psi} \bar\psi + J^a_\eta \eta^a + J^a_{\bar\eta} \bar\eta^a \, .$$

a) Interaction vertices are found from S_{int}: We expand each term in S to find the S_{free} & S_{int}:

* $$\tfrac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} = -\tfrac{1}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \right)\left(\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} + g f^{ade} A^{\mu,d} A^{\nu,e} \right)$$

$$= -\tfrac{1}{2} (\partial_\mu A^a_\nu)(\partial^\mu A^{\nu,a}) + \tfrac{1}{2} (\partial_\mu A^a_\nu)(\partial^\nu A^{\mu,a})$$

$$- g f^{abc} \, \partial_\mu A^a_\nu \, A^{\mu,b} A^{\nu,c} - \tfrac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu,d} A^{\nu,e}$$

* we write the ghost part as:

$$\Rightarrow (\partial^\mu \bar\eta^a)(D^{ab}_\mu \eta^b) = \partial^\mu \big(\bar\eta^a \underbrace{D^{ab}_\mu \eta^b}_{\to 0} \big) - \bar\eta^a \partial^\mu D^{ab}_\mu \eta^b \quad m$$

$$= -\bar\eta^a \partial^\mu D^{ab}_\mu \eta^b$$

$$\Rightarrow -\bar\eta^a \partial^\mu D^{ab}_\mu \eta^b = -\bar\eta^a \partial^\mu \partial_\mu \eta_b + ig\, \bar\eta^a \partial^\mu \left(A^c_\mu (T^c_A)^{ab} \eta^b \right)$$

$$= -\bar\eta^a \delta^{ab} \partial^\mu \partial_\mu \eta_b + ig \left[(\partial^\mu A^c_\mu)(\bar\eta^a (T^c_A)^{ab} \eta^b) + A^c_\mu \bar\eta^a (T^c_A)^{ab} \partial^\mu \eta^b \right]$$

* fermion part:

(minus)

$$\Rightarrow \left(\bar{\Psi}^i (i\gamma^\mu D^{ij}_\mu - m\,\delta^{ij})\psi^j\right) = \bar{\Psi}^i (i\gamma^\mu \partial^{ij}_\mu - m\,\delta^{ij})\psi^j - ig\,\bar{\Psi}^i (i\gamma^\mu A^c_\mu T_F^{ij,c}\psi^j)$$

———— o ————

Now we can write S_{free} & S_{int} as:

$$S_{free} = \int d^4x \Big[-\frac{1}{2}(\partial_\mu A^a_\nu)(\partial^\mu A^{\nu a}) + \frac{1}{2}(\partial_\mu A^a_\nu)(\partial^\nu A^{\mu,a}) - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 - \bar{\eta}^a \delta^{ab}\partial^\mu \partial_\mu \eta$$

$$+ \bar{\Psi}^i (i\gamma^\mu \partial^{ij}_\mu - m\,\delta^{ij})\psi^j \Big]$$

$$= \int d^4x \Big[-\frac{1}{2} A^a_\mu \delta^{ab}\left(g^{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right) A^b_\nu - \bar{\eta}^a \partial^\mu \partial_\mu \delta^{ab}\eta^b$$

$$+ \bar{\Psi}^i (i\gamma^\mu \partial^{ij}_\mu - m\,\delta^{ij})\psi^j \Big]$$

———— o ————

$$S_{interaction} = \int d^4x \Big[-g f^{abc}\partial_\mu A^a_\nu A^{\mu,b} A^{\nu,c} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu,d} A^{\nu,e}$$

$$+ ig(\partial^\mu A^c_\mu)\,\bar{\eta}^a (T^c_A)^{ab}\eta^b + igA^c_\mu\,\bar{\eta}^a (T^c_A)^{ab}\partial^\mu \eta^b$$

$$- ig\,\bar{\Psi}^i (i\gamma^\mu A^c_\mu T_F^{ij,c})\psi^j \Big]$$

To find the vertices, we go to the momentum space first:

$$A^a_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, A^a_\mu(k) \qquad \eta_a(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \eta_a(k)$$

$$\psi^i(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \psi^i(k) \qquad \bar\eta_a(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \bar\eta_a(k)$$

$$\bar\psi^i(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \bar\psi^i(k)$$

S_{int} becomes:

$$S_{int} = \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k_1-k_2-k_3)$$

$$\times ig \Big[\underbrace{k_1^\mu f^{abc} A^a_\nu(k_1) A^b_\mu(k_2) A^{\nu,c}(k_3)}_{1} + (k_2^\mu + k_3^\mu)\, \underbrace{\bar\eta_a(k_1) (T^c_A)^{ab} \eta_b(k_2) A^c_\mu(k_3)}_{2} \Big]$$

$$- ig\, \underbrace{\bar\psi^i(k_1) (i\gamma^\mu A^c_\mu(k_2) T_F^{ij,c}) \psi^j(k_3)}_{4} \Big]$$

$$+ \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^3k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4} (2\pi)^4 \delta^4(k_1-k_2-k_3-k_4)$$

$$g^2 \Big[-\frac{1}{4} f^{abc} f^{ade} \underbrace{A^b_\mu(k_1) A^c_\nu(k_2) A^{\mu,d}(k_3) A^{\nu,d}(k_4)}_{3} \Big]$$

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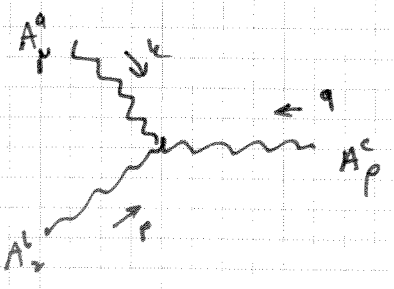
1 Then the vertices are: $(\times(-i))$



$$= \left(g f^{abc} (-k^\nu) g^{\mu\rho} + \text{all permutations} \right) \delta(k+p+q)$$

$$= g f^{abc} \left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-k)^\mu + g^{\mu\rho}(q-k)^\nu \right] \delta(k+p+q)$$

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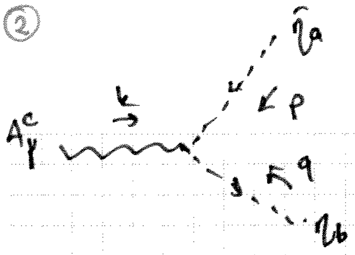
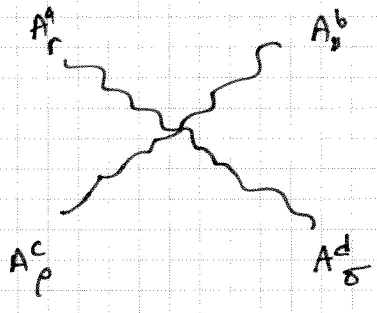
2



$$= -g\, p_\mu\, (T^c_A)^{ab} = -g\, p_\mu\, f^{abc}\, \delta(k+p+q)$$

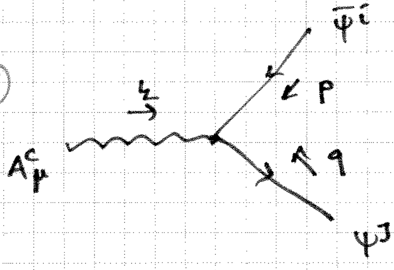
$\uparrow$ (from $\delta(k+p+q)$)

3



$$= ig^2 \Big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big].$$

4



$$= -ig\, \gamma^\mu \left(T^{ij,c}\right)\, \delta(k+p+q)$$

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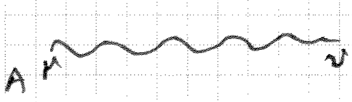
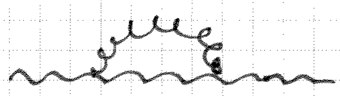
b) The gluon propagator.

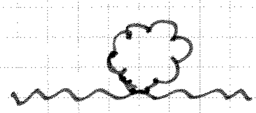


+ $\mathcal{O}(g^2)$



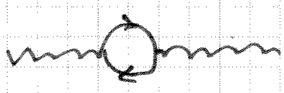
(from 3 gluon vertices)

$\mathcal{O}(g^2)$



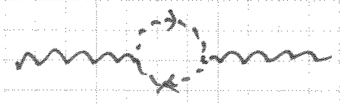
(from 4 gluon vertices

$\mathcal{O}(g^2)$



(from gluon-fermion-fermion vert

+

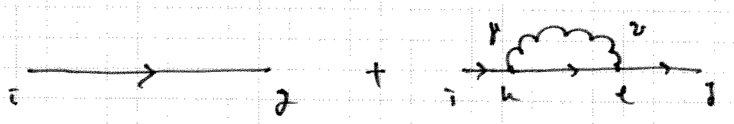


(from gluon-ghost-ghost vertex)

$\mathcal{O}(g^2)$

c) fermion propagator:

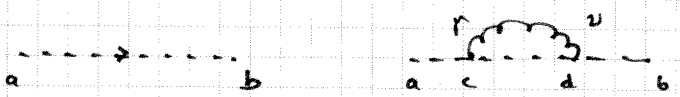
$\mathcal{O}(g^2)$



from (gluon - ferm - ferm vertex)

ghost propagator:

$\mathcal{O}(g^2)$



from (ghost - ghost - gluon vertex)

~~prop~~)