Exercise 1. ***Free particle two-point function I***

Calculate the two-point function $\langle 0, t_b | T(\hat{x}(t)\hat{x}(t')) | 0, t_a \rangle$ for a free particle explicitly

(a) Prove the following integral identities:

$$\int_{-\infty}^{\infty} dx\ x e^{iax^2 + ikx} = -\left(\frac{k}{2a}\right)\sqrt{\frac{i\pi}{a}}\, e^{-\frac{ik^2}{4a}}$$
$$\int_{-\infty}^{\infty} dx\ x^2 e^{iax^2} = -\frac{1}{2ia}\sqrt{\frac{i\pi}{a}}$$

(b) Compute $\langle 0, t_b | \hat{x}(t')\hat{x}(t) | 0, t_a \rangle$ by using the definition: $\hat{x}(t) = \int dx\ x |x, t\rangle\langle x, t|$ together with the explicit expression for the free particle propagator derived in Sheet 1 Exercise 1(c).

Hint. Use the integrals derived in part (a).

(c) Using the definition of the time-ordered product:

$$T(\hat{x}(t)\hat{x}(t')) := \theta(t - t')\hat{x}(t)\hat{x}(t') + \theta(t' - t)\hat{x}(t')\hat{x}(t)$$

calculate $\langle 0, t_b | T(\hat{x}(t)\hat{x}(t')) | 0, t_a \rangle$ by using the expression derived in part (b).

Exercise 2. ***Free particle two-point function II***

Calculate the two-point function $\langle 0, t_b | T(\hat{x}(t)\hat{x}(t')) | 0, t_a \rangle$ for a free particle using the relation:

$$\langle 0, t_b | T(\hat{x}(t)\hat{x}(t')) | 0, t_a \rangle = \left(\frac{\hbar}{i}\right)^2 \left[\frac{\delta^2}{\delta J(t)\delta J(t')}\langle 0, t_b | 0, t_a \rangle_J\right]_{J(t)=J(t')=0}$$

where the free particle generating (partition) function is defined as:

$$\mathcal{Z}[J] := \langle 0, t_b | 0, t_a \rangle_J = \int_{x_a=0}^{x_b=0} \mathcal{D}x\ e^{\frac{i}{\hbar}S[x,J]}$$

where $S[x, J] = \int_{t_a}^{t_b} dt\ \frac{1}{2}m\dot{x}^2 + J(t)x$

(a) By defining $x(t) = \bar{x}(t) + \eta(t)$, where $\bar{x}(t)$ satisfies the equation of motion with boundary conditions $x(t_a) = x_a = x(t_b) = x_b = 0$, and $\eta(t)$ are 'quantum' fluctuations which satisfy $\eta(t_a) = \eta(t_b) = 0$, show that:

$$S[x, J] = S[\bar{x}, J] + S[\eta, 0]$$

Hint. Integrate by parts and use the fact that $\bar{x}(t)$ is a solution to the equation of motion for the action $S[x, J]$.

(b) Using the result of part (a) show that:

$$\mathcal{Z}[J] = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}}\, e^{\frac{i}{\hbar}S[\bar{x},J]}$$

Hint. *Argue that* $\mathcal{D}x = \mathcal{D}\eta$.

(c) Show that one can write:

$$S[\bar{x}, J] = \frac{1}{2m}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\; J(t)G(t,t')J(t')$$

where $G(t,t')$ is the Green's function of the operator $\frac{d^2}{dt^2}$.

Hint. *Integrate by parts and use the relation:* $\bar{x}(t) = \frac{1}{m}\int_{t_a}^{t_b} dt\; G(t,t')J(t')$.

(d) Show that the Green's function $G(t,t')$ with boundary conditions $G(t_a,t') = G(t_b,t') = 0$ is given by:

$$G(t,t') = \theta(t-t')\frac{(t-t_b)(t'-t_a)}{(t_b-t_a)} + \theta(t'-t)\frac{(t'-t_b)(t-t_a)}{(t_b-t_a)}$$

Hint. *Use the defining relation of the Green's function* $\frac{d^2}{dt^2}G(t,t') = \delta(t-t')$ *and solve* $G(t,t')$ *separately in the regions* $t < t'$ *and* $t > t'$ *subject to the boundary conditions. To fix the integration constants assume that* $G(t,t')$ *is continuous at* $t = t'$ *and use the fact that:*
$\lim_{\epsilon\to 0}\left[\frac{dG(t'+\epsilon,t')}{dt} - \frac{dG(t'-\epsilon,t')}{dt}\right] = 1.$

(e) Using the functional definition of the two-point function at the beginning of the question together with the results of parts (b), (c) and (d), calculate an explicit expression for $\langle 0, t_b|T(\hat{x}(t)\hat{x}(t'))|0, t_a\rangle$.

Hint. *You should obtain the same expression as derived in the first exercise!*

Exercise 3. ***Euclidean time and Statistical Mechanics***

Consider the amplitude $A(x_i, x_f, ; T)$ of a free particle. If we continue analytically the time parameter to purely imaginary values by $T \to -i\beta$, with real β (setting $\hbar = 1$), its Schroedinger equation

$$\frac{\partial}{\partial T}A(x_i, x_f, ; T) = -\frac{1}{2m}\frac{\partial^2}{\partial x_f^2}A(x_i, x_f, ; T)$$

with initial conditions $A(x_i, x_f, ; 0) = \delta(x_f - x_i)$ becomes the heat equation

$$\frac{\partial}{\partial \beta}A(x_i, x_f, ; T) = \frac{1}{2m}\frac{\partial^2}{\partial x_f^2}A(x_i, x_f, ; T)$$

The solution, with boundary conditions $A \xrightarrow{\beta\to 0} \delta(x_f - x_i)$, is given by

$$A = \sqrt{\frac{m}{2\pi\beta}}\, e^{-\frac{m(x_f-x_i)^2}{2\beta}}$$

This analytic continuation is called "Wick Rotation". It can be performed directly on the path integral: analytically continuing the time variable as $t \to -i\tau$, the action with "Minkowskian" time (i.e. with a real time t) becomes an "Euclidean" action S_E defined by

$$iS[x] = i\int_0^T dt\,\frac{m}{2}\,\dot{x} \qquad \to \qquad -S_E[x] = -\int_0^\beta d\tau\,\frac{m}{2}\,\dot{x}$$

where in the Euclidean action $\dot{x} = \frac{dx}{d\tau}$, with τ called "Euclidean time". This action is positive definite and the corresponding path integral

$$\int Dx\, e^{-S_E[x]}$$

for a free theory is truly gaussian, with an exponential damping instead of an increasingly rapid phase oscillations. It coincides with the functional integral introduced by Wiener in the 1920's to study Brownian motion and the heat equation.

The Euclidean path integrals are quite useful in statistical mechanics, where β is related to the inverse temperature Θ by $\beta = 1/(k\Theta)$ (k is the Boltzmann's constant). The trace of the evolution operator Z, that can be written using energy eigenstates labeled by n (if the spectrum is discrete) or position eigenstates labeled by q,

$$Z \equiv Tr\, e^{-\frac{i}{\hbar}\hat{H}T} = \sum_n e^{-\frac{i}{\hbar}E_n T} = \int dq\, \langle q|e^{-\frac{i}{\hbar}\hat{H}T}|q\rangle$$

can be Wick rotated with $T \to -i\beta$. Setting again $\hbar = 1$, one obtains the statistical partition function Z_E of the quantum system with hamiltonian $\hat{H}$

$$Z_E \equiv Tr\, e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} = \int dq\, \langle q|e^{-\beta\hat{H}}|q\rangle$$

We can now easily obtain a representation of Z_E in terms of path integrals: perform a Wick rotation of the path integral action, set the initial state (at euclidean time $\tau = 0$) equal to the final state (at Euclidean time$\tau = \beta$), and sum over all possible states. Note that the paths are now closed paths, as $q(0) = q(\beta)$, and the partition function becomes

$$Z_E \equiv Tr\, e^{-\beta\hat{H}} = \int_{PBC} Dq\, e^{-S_E[q]}$$

where PBC stands for "periodic boundary conditions", indicating the sum over all paths that close on themselves in an Euclidean time β.

Moreover, if you consider the statistical partition function in the limit of vanishing temperature ($\Theta \to 0$), or equivalently for an infinite Euclidean propagation time ($\beta \to \infty$), it becomes simply

$$Z_E \equiv Tr\, e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} \xrightarrow{\beta\to 0} e^{-\beta E_0} + \text{ subleading terms}$$

This is true even in the presence of a source J, if one assumes that the source is non-vanishing only in a finite interval of time: the remaining infinite time is sufficient to project the operator $e^{-\beta\hat{H}}$ onto the ground state. This allows us to rewrite the generating functional $Z[J]$ in the euclidean case in a simpler way, justifying the dropping of boundary terms in the integration by parts.

The reason why we mention this relation between Euclidean and Minkowskian path integrals is deeper. Often, even if one is interested in the theory with a real time, one works with the

Euclidean theory, where path integral convergence is more easily kept under control. Only at the very end one performs the inverse Wick rotation to read off the result for the Minkowskian theory. Moreover, path integrals in Euclidean times are mathematically better defined (one may develop a mathematically well defined measure theory on the space of functions), at least for quadratic actions and perturbations thereof, while path integral with a Minkowskian time are more delicate. The Wick rotation suggests a way of defining the path integral in real time starting from the one with Euclidean time.

Let's now practise a bit with Euclidean theories.

(a) By performing a Wick rotation $t \to -i\tau$, show that the Green's function equation for the Euclidean simple harmonic oscillator operator is given by:

$$\left(-\frac{d^2}{d\tau^2} + \omega^2\right) G_E(\tau, \tau') = \delta(\tau - \tau')$$

(b) Show that in the case where the integration range of the action integral is extended such that $\tau \in (-\infty, \infty)$, the Euclidean Green's function has the explicit form:

$$G_E(\tau, \tau') = \frac{1}{2\omega} e^{-\omega(\tau-\tau')} \theta(\tau - \tau') + \frac{1}{2\omega} e^{-\omega(\tau'-\tau)} \theta(\tau' - \tau) = \frac{1}{2\omega} e^{-\omega|\tau-\tau'|}$$

Hint. Extending the integration range of τ in this way allows one to solve the Green's function equation using Fourier transform techniques. First, take the Fourier transform $\mathcal{F}$ of the Green's function equation in part (a) and determine an algebraic equation for $\mathcal{F}[G_E(\tau, \tau')]$. Then perform the inverse Fourier transform $\mathcal{F}^{-1}$ and use contour integration to explicitly solve the integral equation for $G_E(\tau, \tau')$.

(c) Consider the previous Euclidean Green's function obtained before preforming the integral. How is it related to the one in the usual Minkowski space? Did you use any prescription to evaluate the integral? Which is the relation between different boundary conditions (that would lead, for instance, to retarded or advanced Green functions) and the corresponding implemented prescriptions for performing the integrals? Can we always use the Wick rotation to pass from Euclidean to Minkowskian propagators?

References

[1] Introduction to the exercise adapted from *http://www-th.bo.infn.it/people/bastianelli/FT-1-ch2.pdf*, (F. Bastianelli).