Exercise 1. ***Renormalization in QCD at one loop: The Vertices***

In this series we will renormalise the vertices of QCD at one-loop. Here you find all the results of the renormalisation in the $\overline{MS}$. Try to do some of these integrals by your own (you don't have to compute all of them, but try to become confident with these kind of calculations).

We define the renormalised quantities as

$$\begin{array}{ll} A^a_{\mu,0} = \sqrt{Z_A}\, A^a_{\mu,R} & g_0 = Z_g\, g_R \\ \eta_{a,0} = \sqrt{\tilde{Z}_3}\, \eta_{a,R} & m_0 = Z_m\, m_R \\ \psi_0 = \sqrt{Z_2}\, \psi_R & \xi_0 = Z_A\, \xi_R \end{array}$$

(a) Ghost-gluon vertex

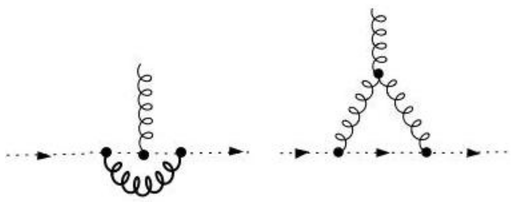

Figure 1: One loop ghost-gluon verteces

$$\begin{aligned} I &= -i\, g\, p_\mu\, (-i\, f_{abc}) \left(\tilde{Z}_1 + \frac{\alpha_s}{4\pi} N_C\, (4\pi)^\varepsilon\, \Gamma(\varepsilon) + finite \right) && (1) \\ \tilde{Z}_1 &= Z_g\, \tilde{Z}_3\, Z_A^{1/2} \\ &= 1 - \frac{\alpha_s}{4\pi} N_C\, (4\pi)^\varepsilon\, \Gamma(\varepsilon) && (2) \end{aligned}$$

(b) Quark-gluon vertex

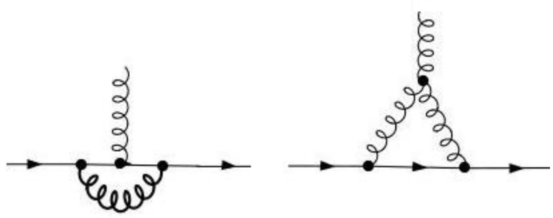

Figure 2: One loop quark-gluon verteces

$$\begin{aligned} I &= -i\, g\, \gamma_\mu\, t^a_{ij} \left(Z_{1F} + \frac{\alpha_s}{4\pi} (N_C + C_F)\, (4\pi)^\varepsilon\, \Gamma(\varepsilon) + finite \right) && (3) \\ Z_{1F} &= Z_g\, Z_2\, Z_A^{1/2} \\ &= 1 - \frac{\alpha_s}{4\pi} (N_C + C_F)\, (4\pi)^\varepsilon\, \Gamma(\varepsilon) && (4) \end{aligned}$$

(c) 3 gluon-gluon vertex

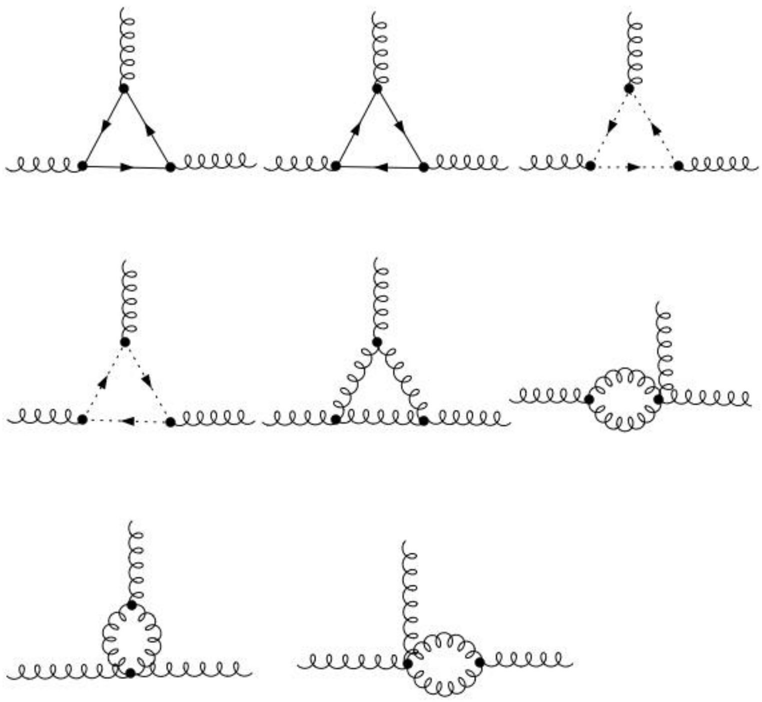

Figure 3: One loop 3 gluon-gluon verteces

$$
\begin{aligned}
I &= -i\,g\,f^{abc}\,V_{\mu\nu\sigma}(p_1,p_2,p_3)\left(Z_1+\frac{\alpha_s}{4\pi}\left(N_C\left(-\frac{2}{3}\right)+\frac{4}{3}\,\frac{1}{2}n_f\right)(4\pi)^{\varepsilon}\,\Gamma(\varepsilon)+finite\right) && (5)\\
Z_1 &= Z_g\,Z_A^{3/2}\\
&= 1-\frac{\alpha_s}{4\pi}\left(N_C\left(-\frac{2}{3}\right)+\frac{4}{3}\,\frac{1}{2}n_f\right)(4\pi)^{\varepsilon}\,\Gamma(\varepsilon) && (6)
\end{aligned}
$$

(d) 4 gluon-gluon vertex

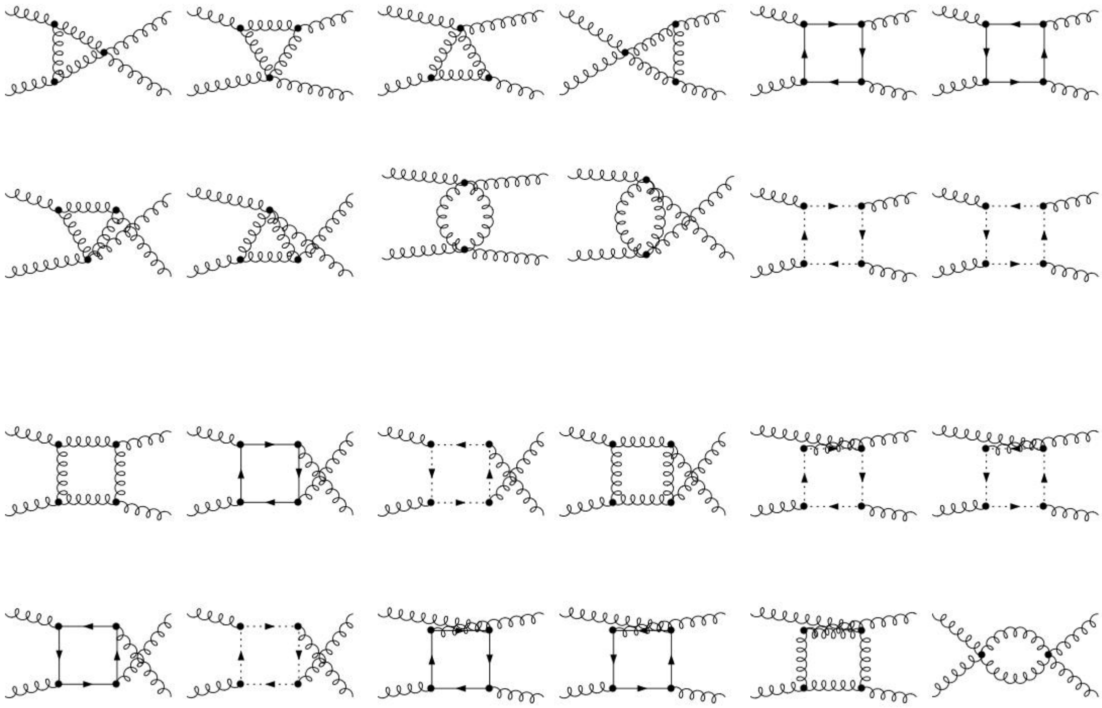

Figure 4: One loop 4 gluon-gluon verteces

$$
\begin{aligned}
I &= -i\, g^2\, V^{abcd}_{\mu\nu\rho\sigma} \left(Z_4 + \frac{\alpha_s}{4\pi} \left(\frac{N_C}{3} + \frac{4}{3}\,\frac{1}{2} n_f \right) (4\pi)^{\varepsilon}\, \Gamma(\varepsilon) + finite \right) && (7) \\
Z_4 &= Z_g^2\, Z_A^2 \\
&= 1 - \frac{\alpha_s}{4\pi} \left(\frac{N_C}{3} + \frac{4}{3}\,\frac{1}{2} n_f \right) (4\pi)^{\varepsilon}\, \Gamma(\varepsilon) && (8)
\end{aligned}
$$

Now define $\Delta_\varepsilon \equiv \frac{\alpha_s}{4\pi}\,(4\pi)^{\varepsilon}\,\Gamma(\varepsilon)$. We have

$$Z_A = 1 - \left[\frac{2}{3}n_f - \frac{5}{3}N_C\right] \Delta_\varepsilon \tag{9}$$

$$\tilde{Z}_1 + \frac{N_C}{2} \Delta_\varepsilon \tag{10}$$

$$Z_2 = 1 - C_F \Delta_\varepsilon \tag{11}$$

$$Z_m = 1 - 3C_F \Delta_\varepsilon \tag{12}$$

$$Z_1 = 1 - \frac{\alpha_s}{4\pi}\left(N_C\left(-\frac{2}{3}\right) + \frac{4}{3}\frac{1}{2}n_f\right) \Delta_\varepsilon \tag{13}$$

$$\tilde{Z}_1 = 1 - \frac{N_C}{2} \Delta_\varepsilon \tag{14}$$

$$Z_{1F} = 1 - \frac{\alpha_s}{4\pi}(N_C + C_F) \Delta_\varepsilon \tag{15}$$

$$Z_4 = 1 - \frac{\alpha_s}{4\pi}\left(\frac{N_C}{3} + \frac{4}{3}\frac{1}{2}n_f\right) \Delta_\varepsilon \tag{16}$$

$$\frac{Z_1}{Z_A} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} = \frac{Z_4}{Z_1} = Z_g\, Z_A^{1/2} = 1 - N_C \Delta_\varepsilon \tag{17}$$

Exercise 2. ***Colour Factors***

It is common to adopt a normalisation convention based on the trace of products for the matrices t^a belonging to an irreducible representation. In particular, for the $SU(N_C)$ Lie Algebra, one usually uses

$$Tr(t^a\, t^b) = T_F\, \delta^{ab}, \qquad T_F = \frac{1}{2}$$

The matrices t^a are hermitian, therefore there are $N_C^2 - 1$ generators which form a basis for this space of hermitian matrices. We then have that their commutators are i times hermitian traceless matrices, i.e.

$$[t^a,\, t^b] = i\, f^{abc}\, t^c, \qquad i\, f^{abc} = \frac{1}{T_F}\, Tr\left([t^a,\, t^b]\, t^c\right)$$

where f^{abc} are real constants. We also know that actually they are the generators in the adjoint representation:

$$(t^c)^{ab} = i\, f^{acb}$$

In order to calculate the colour factors, we know that

$$\begin{aligned}
&Tr \text{ (fermion loop) } = N_C \\
&Tr(t^a) = 0 \\
&Tr(t^a\, t^b) = T_F\, \delta^{ab} \\
&t^a\, t^a = C_F, \qquad C_F = \frac{1}{2}\frac{N_C^2 - 1}{N_C}
\end{aligned} \tag{18}$$

Show that this other useful identity holds

$$(t^a)^i_j\, (t^a)^k_l = T_F\left[\delta^i_l\, \delta^k_j - \frac{1}{N_C}\, \delta^i_j\, \delta^k_l\right] \tag{19}$$

Hint. *First notice that the general form of this expression is*

$$(t^a)^i_j\,(t^a)^k_l = a\,\left[\delta^i_l\,\delta^k_j - b\,\delta^i_j\,\delta^k_l\right]$$

Then, multiply it once by δ^j_i *and the other time by* $(t^b)^j_i$ *in order to find who* b *and* a *are.*