Ex. 1

a) For the second term of the expansion, using again the completeness relation

$$\int dx \; |x\rangle\langle x| = \mathbb{1}$$

we obtain

$$A^2_\phi = (-ig)^2 \int dx_1 \int dx_2 \; \phi(x_1)\,\phi(x_2)\;(-1)\,\mathrm{Tr}\left[\Delta_F(x_2-x_1)\cdot\Delta_F(x_1-x_2)\right]$$

b) Introducing the centre of mass and relative coordinates

$$\bar{x} = \frac{x_1+x_2}{2} \qquad x = x_1 - x_2 \qquad \frac{\partial(x_1,x_2)}{\partial(\bar{x},x)} = 1$$

we have

$$A^2_\phi = g^2 \int d\bar{x}\int dx \;\phi\left(\bar{x}+\frac{x}{2}\right)\phi\left(\bar{x}-\frac{x}{2}\right)\,\mathrm{tr}\left[\Delta_F(-x)\,\Delta_F(x)\right]$$

$$= g^2 \int d\bar{x}\int dx \int\frac{dl}{(2\pi)^4}\,\tilde{\phi}(l)\int\frac{du}{(2\pi)^4}\,\tilde{\phi}(u)\;\cdot$$

$$\cdot\, e^{-i\bar{x}\cdot(l+u)\,-\,ix\cdot(l-u)/2}\;\cdot$$

$$\cdot\int\frac{dp}{(2\pi)^4}\int\frac{dq}{(2\pi)^4}\;e^{i(p-q)\cdot x}\;\mathrm{tr}\left[\frac{i}{\not{p}-M+i\varepsilon}\;\frac{i}{\not{q}-M+i\varepsilon}\right]$$

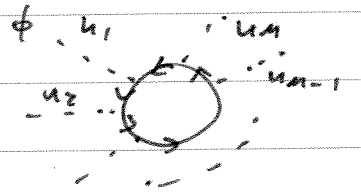
$$= -g^2\int\frac{du}{(2\pi)^4}\,\tilde{\phi}(u)\,\tilde{\phi}(-u)\int\frac{dp}{(2\pi)^4}\int\frac{dq}{(2\pi)^4}\;\cdot$$

$$\cdot\,\frac{\mathrm{tr}\left[(\not{p}+M)(\not{q}+M)\right]}{(p^2-M^2+i\varepsilon)(q^2-M^2+i\varepsilon)}\;(2\pi)^4\,\delta(u+p-q)$$

$$= (-ig)^2 \int \frac{dk}{(2\pi)^4} \; \tilde{\phi}(k)\, \tilde{\phi}(-k) \; (-1) \int \frac{dp}{(2\pi)^4} \, \mathrm{tr}\left[\frac{i}{\not{p} - M + i\varepsilon} \; \frac{i}{\not{p} + \not{k} - M + i\varepsilon} \right.$$

where the second integral represents a fermion loop with two propagators :

c) From the general term A^n_ϕ we deduce that it corresponds to a fermion loop with n- external legs associated to the scalar field vertices :



Notice that from here we also see that for every fermion loop we have to multiply it by a factor (-1). and it always involves a trace operation over the spinor indices.

Ex 2

[redacted] Dual field stength tensors

a) By definition, the field strength tensor is antisymmetric

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We have therefore

$$\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma - \epsilon^{\mu\nu\rho\sigma}\partial_\sigma A_\rho = \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma + \epsilon^{\mu\nu\sigma\rho}\partial_\sigma A_\rho = 2\epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma$$

and similarly $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} = 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu$.

Hence

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 2\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_\nu\right)\left(\partial_\rho A_\sigma\right)$$

On the other side, we have

$$K^\mu = 2\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma \quad \Rightarrow \quad \partial_\mu K^\mu = 2\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_\nu\right)\left(\partial_\rho A_\sigma\right) + 2\epsilon^{\mu\nu\rho\sigma} A_\nu \left(\partial_\mu \partial_\rho A_\sigma\right)$$

In the last equality the partial derivatives commute ($\partial_\mu\partial_\rho = \partial_\rho\partial_\mu$), thus the last term is symmetric in μ and ρ and its contraction with the ϵ tensor vanishes.

We obtain finally

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_\nu\right)\left(\partial_\rho A_\sigma\right) = \partial_\mu K^\mu$$

b) Like in the abelian case, the field strength tensor is antisymmetric

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

But in this case the A's are matrices and do not commute.

One can apply the same trick as in part a) and show that

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma} &= \epsilon^{\mu\nu\rho\sigma}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho - igA_\rho A_\sigma + igA_\sigma A_\rho\right) \\
&= \epsilon^{\mu\nu\rho\sigma}\left(\partial_\rho A_\sigma - igA_\rho A_\sigma\right) + \epsilon^{\mu\nu\sigma\rho}\left(\partial_\sigma A_\rho - igA_\sigma A_\rho\right) \\
&= 2\epsilon^{\mu\nu\rho\sigma}\left(\partial_\rho A_\sigma - igA_\rho A_\sigma\right)
\end{aligned}$$

and $\epsilon^{\mu\nu\rho\sigma}G_{\mu\nu} = 2\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_\nu - igA_\mu A_\nu\right)$, so that

$$\begin{aligned}
\operatorname{Tr} G_{\mu\nu}\tilde{G}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr} G_{\mu\nu}G_{\rho\sigma} = 2\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left(\partial_\mu A_\nu - igA_\mu A_\nu\right)\left(\partial_\rho A_\sigma - igA_\rho A_\sigma\right) \\
&= 2\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left[\left(\partial_\mu A_\nu\right)\left(\partial_\rho A_\sigma\right) - ig\left(\partial_\mu A_\nu\right)A_\rho A_\sigma - ig\left(\partial_\rho A_\sigma\right)A_\mu A_\nu - g^2 A_\mu A_\nu A_\rho A_\sigma\right] \\
&= 2\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left[\left(\partial_\mu A_\nu\right)\left(\partial_\rho A_\sigma\right) - 2ig\left(\partial_\mu A_\nu\right)A_\rho A_\sigma\right]
\end{aligned}$$

From the first to the second line we have used the cyclicity of the trace to move $A_\mu A_\nu$ to the right in the third term, and then permuting the indices one can see that the second and third terms are the same. The term with 4 A's vanishes since

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr} A_\mu A_\nu A_\rho A_\sigma &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\operatorname{Tr} A_\mu A_\nu A_\rho A_\sigma + \operatorname{Tr} A_\nu A_\rho A_\sigma A_\mu\right) \\
&= \frac{1}{2}\left(\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr} A_\mu A_\nu A_\rho A_\sigma - \epsilon^{\nu\rho\sigma\mu}\operatorname{Tr} A_\nu A_\rho A_\sigma A_\mu\right) = 0
\end{aligned}$$

Notice that while the completely antisymmetric tensor with 3 indices ϵ^{ijk} is cyclic, the 4-indices tensor $\epsilon^{\mu\nu\rho\sigma}$ is not!

K^μ can be written as

$$K^\mu = 2\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left[\left(\partial_\nu A_\rho\right)A_\sigma - \frac{2}{3}igA_\nu A_\rho A_\sigma\right]$$

so that

$$\begin{aligned}
\partial_\mu K^\mu = \; & 2\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left[\left(\partial_\mu\partial_\nu A_\rho\right)A_\sigma + \left(\partial_\nu A_\rho\right)\left(\partial_\mu A_\sigma\right)\right. \\
& \left. -\frac{2}{3}ig\left(\partial_\mu A_\nu\right)A_\rho A_\sigma - \frac{2}{3}ig\left(\partial_\mu A_\rho\right)A_\sigma A_\nu - \frac{2}{3}ig\left(\partial_\mu A_\sigma\right)A_\nu A_\rho\right]
\end{aligned}$$

(we have used the cyclicity of the trace again)

The first term in the trace vanishes as above because the partial derivatives commute. Permuting the indices, the second term becomes $(\partial_\mu A_\nu)(\partial_\rho A_\sigma)$, and the three last terms are equal to each other, so that finally

$$\partial_\mu K^\mu = 2\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left[\left(\partial_\mu A_\nu\right)\left(\partial_\rho A_\sigma\right) - 2ig\left(\partial_\mu A_\nu\right)A_\rho A_\sigma\right] = \operatorname{Tr} G_{\mu\nu}\tilde{G}^{\mu\nu}$$

Exercise [redacted]3 Path integral in gauge theories

All we need to show is that the Jacobian determinant $\det\left(\frac{\delta A'^a_\mu}{\delta A^b_\mu}\right)$ is equal to 1. In order to show this, we need to know A'^a_μ, i.e. the gauge transformation of the uncontracted field. This can be derived from the transformation law for A_μ, retaining only the lowest order in g:

$$\begin{aligned}
A'^a_\mu T^a &= A'_\mu = e^{-ig\theta^a T^a} A^c_\mu T^c e^{ig\theta^b T^b} - \frac{i}{g} e^{-ig\theta^a T^a} \partial_\mu e^{ig\theta^b T^b} \\
&\approx (1 - ig\theta^a T^a) A^c_\mu T^c (1 + ig\theta^b T^b) - \frac{i}{g}(1 - ig\theta^a T^a)(ig\partial_\mu \theta^b T^b) \\
&= A^c_\mu T^c + igA^c_\mu(\theta^a T^a T^c - \theta^b T^c T^b) + \partial_\mu \theta^b T^b + ig\theta^a \partial_\mu \theta^b T^a T^b \\
&= A^c_\mu T^c + igA^c_\mu \theta^a (if^{acb} T^b) + \partial_\mu \theta^b T^b + ig\theta^a \partial_\mu \theta^b T^a T^b \\
&= \left(A^a_\mu + gA^a_\mu \theta^b f^{abc} + \partial_\mu \theta^a + ig\theta^b \partial_\mu \theta^a T^b\right) T^a
\end{aligned}$$

where we've busily renamed and reshuffled indices. Thus, the transformation for A'^a_μ reads:

$$A'^a_\mu = A^a_\mu + gf^{abc}\theta^c A^b_\mu + \partial_\mu \theta^a + ig\theta^b \partial_\mu \theta^a T^b$$

Calculating the Jacobian is a piece of cake now:

$$J^{ab} = \frac{\delta A'^a_\mu}{\delta A^b_\mu} = \delta^{ab} + gf^{abc}\theta^c$$

And thus, using that the structure constants f^{abc} are antisymmetric in every pair of indices (and therefore the matrix $(f^c)_{ab}$ has zeroes on the diagonal):

$$\det(J) = \det(1 + gf^c\theta^c) = \exp \operatorname{Tr} \log(1 + gf^c\theta^c) \approx \exp \operatorname{Tr}(gf^c\theta^c) = exp(0) = 1$$

We have proven that the Jacobian determinant is unity up to $\mathcal{O}(g^2)$, and thus the path integral measure is invariant under gauge transformations.

QUESTION 4 SOLUTION

$$\delta S_{YM} = \frac{1}{2}\int d^4x \; 2\cdot \mathrm{Tr}\left[\delta F_{\mu\nu} F^{\mu\nu}\right]$$

Now $\delta F^a_{\mu\nu} = \delta\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig f^{abc} A^b_\mu A^c_\nu\right)$

$$= \partial_\mu(\delta A^a_\nu) - \partial_\nu(\delta A^a_\mu) + ig f^{abc}\left[\delta A^b_\mu A^c_\nu + A^b_\mu \delta A^c_\nu\right]$$

$$\Rightarrow \delta S_{YM} = \int d^4x \; \mathrm{Tr}\left[F^{\mu\nu}\left\{\partial_\mu(\delta A^a_\nu) - \partial_\nu(\delta A^a_\mu) + ig f^{abc}\left[\delta A^b_\mu A^c_\nu + A^b_\mu \delta A^c_\nu\right]\right\}\right]$$

$$= \int d^4x \; \mathrm{Tr}\left[2F^{\mu\nu}\partial_\mu(\delta A_\nu) + 2F^{\mu\nu}\left(ig f^{abc} A^b_\mu \delta A^c_\nu\right)\right]$$

using $F^{\mu\nu} = -F^{\nu\mu}$

and: $f^{abc} = -f^{acb}$.

$$= 2\int d^4x \; \mathrm{Tr}\left[F^{\mu\nu}\partial_\mu(\delta A_\nu)\right] + 2\int d^4x \; \mathrm{Tr}\left(F^{\mu\nu}\left(ig f^{abc} A^b_\mu \delta A^c_\nu\right)\right)$$

$$= 2\int d^4x \; \partial_\mu\left\{\mathrm{Tr}\left[F^{\mu\nu}\delta A_\nu\right]\right\} - 2\int d^4x \; \mathrm{Tr}\left[(\partial_\mu F^{\mu\nu})\delta A_\nu\right]$$

$$+ 2\int d^4x \; \mathrm{Tr}\left[F^{\mu\nu}\left(ig f^{abc} A^b_\mu \delta A^c_\nu\right)\right]$$

$$= -2\int d^4x \; \mathrm{Tr}\left[(D_\mu F^{\mu\nu})\delta A_\nu\right] + 2\int d^4x \; \partial_\mu\left\{\mathrm{Tr}\left[F^{\mu\nu}\delta A_\nu\right]\right\}.$$

using anti-symmetry of f^{abc}

and: $D_\mu F^{\mu\nu} := \partial_\mu F^{\mu\nu} - ig\left[A_\mu, F^{\mu\nu}\right]$

$\Rightarrow D_\mu F^{\mu\nu,a} = \partial_\mu F^{\mu\nu,a} - ig f^{abc} A^b_\mu F^{\mu\nu,c}$

Now one assumes variations δA with compact support

$\Rightarrow \delta A\big|_{\partial\mathbb{R}^{1,3}} = 0$

$\Rightarrow$ the 2nd term vanishes after using the divergence theorem.

- Hence: $\delta S_{YM} = 0 \Rightarrow \underline{D_\mu F^{\mu\nu,a} = 0}$, for arbitrary δA.