Exercise 1. ***1-loop correction to the effective potential***

a) Plugging in $V''(\phi_{cl}) = m^2 - \frac{\lambda}{2}\phi_{cl}^2$ we have:

$$V^{(1)} = -\frac{1}{VT}\frac{1}{2}\mathrm{Tr}\ln(\partial^2 + m^2 + \frac{\lambda}{2}\phi_{cl}^2) \tag{1}$$

$$= -\frac{1}{VT}\frac{1}{2}\mathrm{Tr}\ln\left((\partial^2 + m^2)\left(1 + \frac{\lambda}{2}(\partial^2 + m^2)^{-1}\phi_{cl}^2\right)\right) \tag{2}$$

$$= -\frac{1}{VT}\frac{1}{2}\left(\mathrm{Tr}\ln(\partial^2 + m^2) + \mathrm{Tr}\ln\left(1 + \frac{\lambda}{2}(\partial^2 + m^2)^{-1}\phi_{cl}^2\right)\right) \tag{3}$$

The first term is a constant which we can drop. We can evaluate the second term by expanding the log:

$$\ln(1-x) = -\sum_n \frac{x^n}{n} \tag{4}$$

So that the second term becomes:

$$\mathrm{Tr}\ln\left(1 + \frac{\lambda}{2}(\partial^2 + m^2)^{-1}\phi_{cl}^2\right) = -\sum_n \frac{1}{n}\mathrm{Tr}\left(-\frac{\lambda}{2}(\partial^2 + m^2)^{-1}\phi_{cl}^2\right)^n \tag{5}$$

We can evaluate the trace now. After replacing the operator $\partial_\mu \to \hat{p}_\mu / i$ we have:

$$\mathrm{Tr}\left((\partial^2 + m^2)\right) = \int dx \langle x|(-\hat{p}^2 + m^2)|x\rangle \tag{6}$$

$$= \int dx \int \frac{d^dk}{(2\pi)^d}\langle x|(-\hat{p}^2 + m^2)|k\rangle\langle k|x\rangle \tag{7}$$

$$= \int dx \int \frac{d^dk}{(2\pi)^d}(-k^2 + m^2)\langle x|k\rangle\langle k|x\rangle \tag{8}$$

$$= VT \int \frac{d^dk}{(2\pi)^d}(-k^2 + m^2) \tag{9}$$

where $\langle x|k\rangle = e^{ik\cdot x}$ is used. With this result now we have:

$$\mathrm{Tr}\left(-\frac{\lambda}{2}(\partial^2 + m^2)^{-1}\phi_{cl}^2\right)^n = \left(-\frac{\lambda}{2}\phi_{cl}^2\right)^n (VT)\int \frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2 + m^2)^n} \tag{10}$$

Therefore $V^{(1)}$ becomes:

$$V^{(1)} = \sum_n \frac{1}{2n}\int \frac{d^dk}{(2\pi)^d}\left(\frac{\lambda}{2}\phi_{cl}^2\right)^n \frac{1}{(-k^2 + m^2)^n} \tag{11}$$

b) For the d-dimensional integration we can apply the usual formula:

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2 - m^2)^n} = \frac{i(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n - d/2)}{\Gamma(n)}(m^2)^{d/2-n} \tag{12}$$

(can be found e.g. in Peskin Schroeder, p. 807).

Plugging it in the potential, we obtain:

$$V^{(1)} = i\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\lambda\phi^2}{2}\right)^n \frac{i(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}(m^2)^{d/2-n} \tag{13}$$

$$= -\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\sum_{n=1}^{\infty}\frac{\Gamma(n-d/2)}{n(n-1)!}\left(-\frac{\lambda\phi^2}{2m^2}\right)^n \tag{14}$$

$$= -\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\Gamma(-d/2)\sum_{n=1}^{\infty}\frac{(-d/2,n)}{n!}\left(-\frac{\lambda\phi^2}{2m^2}\right)^n \tag{15}$$

where in the second equality we have used $\Gamma(n) = (n-1)!$ for a positive integer n. Identifying $w = \frac{-\lambda\phi^2}{2m^2}$, this is exactly the equation we wanted. Now using the identity:

$$\sum_{n=0}^{\infty}\frac{(-d/2,n)}{n!}w^n = 1 + \sum_{n=1}^{\infty}\frac{(-d/2,n)}{n!}w^n = (1-w)^{d/2} \tag{16}$$

we have

$$V^{(1)} = -\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\Gamma(-d/2)\left[\left(1+\frac{\lambda\phi^2}{2m^2}\right)^{d/2} - 1\right] \tag{17}$$

We make the substitution $d = 4 - 2\epsilon$ and using the formula

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} - \gamma + \left(\sum_{i=1}^{n}\frac{1}{i}\right) + \mathcal{O}(\epsilon)\right] \tag{18}$$

for an integer $n \geq 0$, and $x^\epsilon = e^{\epsilon\log x} = 1 + \epsilon\log x + \mathcal{O}(\epsilon)$, we get finally

$$V^{(1)} = -\frac{1}{4}\frac{m^4}{(4\pi)^2}\frac{1}{\epsilon}\left[\left(1+\frac{\lambda\phi^2}{2m^2}\right)^2 - 1\right] + \mathcal{O}(\epsilon^0) \tag{19}$$

$$= -\frac{1}{\epsilon}\frac{\lambda}{(4\pi)^2}\left(\frac{m^2\phi^2}{4} + \frac{\lambda\phi^4}{16}\right) + \text{finite} \tag{20}$$

Sheet 8 – Exercise 2

(a) $\langle 0|\delta_B O|0\rangle = \langle 0|[iQ_B, O]_\pm|0\rangle = \underbrace{\langle 0|iQ_B}_{\propto Q_B^\dagger|0\rangle = Q_B|0\rangle = 0} O \pm O \underbrace{iQ_B|0\rangle}_{=0} = 0$

$=0$ since $|0\rangle$ is physical & using $Q_B|\text{phys}\rangle = 0$

since Q_B is Hermitian

(b) $$S[J_I, K_I] = \int d^4x \Big[J^{\mu a} A_\mu^a + J_\psi^i \psi^i + J_\eta^a \eta^a + J_{\bar\eta}^a \bar\eta^a + J_w^a w^a$$

$$- \underbrace{K^{\mu a}\frac{\theta}{g}(D_\mu\eta)^a}_{:= K_A^a(\delta_B A_\mu^a)} - ig \underbrace{K_\psi^i \theta \eta^a (T^a)_{ij}\psi^j}_{:= K_\psi^i(\delta_B\psi^i)} + \frac{1}{2}\underbrace{K_\eta^a \theta f^{abc}\eta^b\eta^c}_{= K_\eta^a(\delta_B\eta^a)} \Big]$$

call the sum of these $\delta_B \alpha$

$$\Rightarrow \delta_B[T\{e^{iS}\}] = T\{(i\delta_B S)e^{iS}\} \quad (\text{where say } \delta_B(J^{\mu a}A_\mu^a) = J^{\mu a}\delta_B A_\mu^a$$
because currents are BRST inv.)

$$= i\int d^4x\, T\Big\{\Big[J^{\mu a}\delta_B A_\mu^a + J_\psi^i \delta_B\psi^i + J_\eta^a \delta_B\eta^a + J_{\bar\eta}^a \delta_B\bar\eta^a$$

$$+ J_w^a \underbrace{\delta_B w^a}_{=0} + \underbrace{\delta_B(\delta_B\alpha)}_{=0 \text{ as } \delta_B^2 = 0}\Big] e^{iS}\Big\}$$

Then inserting this expression in $\langle 0|\cdots|0\rangle$ and using the result of part (a) gives the final expression.

(c) One has: $$i\int d^4x \langle 0|T\Big\{\Big[-J^{\mu a}\frac{\theta}{g}(D_\mu\eta)^a - ig J_\psi^i \theta\eta^a(T^a)_{ij}\psi^j \qquad (+)$$
$$+ \frac{1}{2}J_\eta^a \theta f^{abc}\eta^b\eta^c + J_{\bar\eta}^a \frac{\theta}{g} w^a\Big] e^{iS}\Big\}|0\rangle = 0$$

For the current terms one ~~[illegible]~~ can replace them by: $$\frac{\delta\Gamma}{\delta\Phi_I} = \begin{cases} -J_I, & \text{for bosonic } \Phi_I \\ +J_I, & \text{for fermionic } \Phi_I \end{cases} \quad (*)$$

which is derived by: $$\frac{\delta}{\delta\Phi_I}\Big\{W - \sum_{I'} J_{I'}\Phi_{I'}\Big\} = -\sum_{I'}\frac{\delta}{\delta\Phi_I} J_{I'}\Phi_{I'}$$

$\frac{\delta W}{\delta\Phi_I} = 0$ since J & K are treated as indept of Φ_I

swapping these gives a $-$ve sign when Φ_I is fermionic

$$= (*)$$

And one uses: $\dfrac{\delta\Gamma}{\delta K} = \dfrac{\delta W}{\delta K} + \underbrace{\sum_I \dfrac{\delta W}{\delta J_I}\dfrac{\delta J_I}{\delta K} - \sum_I \dfrac{\delta J_I}{\delta K}\Phi_I}_{=0 \text{ since } \Phi_I := \frac{\delta W}{\delta J_I}} = \dfrac{\delta W}{\delta K}$

Now: $e^{iW} = \langle 0 | T\{e^{iS}\} | 0\rangle$ (eqn (21)) from the discussion in the text.

$$\Rightarrow \frac{\delta}{\delta K} e^{iW} = i\frac{\delta W}{\delta K} e^{iW} = \langle 0 | T\{ \frac{\delta S}{\delta K} e^{iS} \} | 0\rangle$$

$$\Rightarrow \frac{\delta W}{\delta K} = \frac{\delta \Gamma}{\delta K} = \langle 0 | T\{\tfrac{\delta S}{\delta K} e^{iS}\} | 0\rangle \times (-ie^{-iW})$$

this is equal to the coefficients in front of the currents J in the expression (+) at the start of this part of the question (on the other page).

Using this one can now re-write expression (+) as:

$$\underbrace{i\int d^4x \left[\frac{\delta\Gamma}{\delta K^{\mu a}} \frac{\delta\Gamma}{\delta A^a_\mu} + \frac{\delta\Gamma}{\delta K^i_\phi}\frac{\delta\Gamma}{\delta \phi^i} + \frac{\delta\Gamma}{\delta K^a_\eta}\frac{\delta\Gamma}{\delta \eta^a} + \frac{\theta}{g} w^a \frac{\delta\Gamma}{\delta \bar\eta^a}\right]}(-ie^{-iW}) = 0 .$$

since e^{-iW} is in general non-vanishing $\Rightarrow$ this must vanish, which is the final result.

<u>Note</u>: $-$ve signs are obtained if one swaps an anti-commuting current with θ.