Ex1 BRST transformation: Serco T

↳ $$\delta_\theta A^a_\mu = -\frac{\theta}{g} D^{ab}_\mu \eta^b ; \qquad D^{ab}_\mu = \left(\delta^{ab}\partial_\mu - g f^{abc} A^c_\mu\right)$$

$$\delta_{\theta_1}\delta_{\theta_2} A^a_\mu = -\frac{\theta_2}{g}\Big(D^{ab}_\mu (\delta_{\theta_1}\eta^b) - g f^{abc} \underbrace{(\delta_{\theta_1} A^c_\mu)}_{-\frac{\theta_1}{g} D^{cd}_\mu \eta^d} \eta^b \Big)$$

$$= -\frac{\theta_2}{g}\Big(D^{ab}_\mu (\delta_{\theta_1}\eta^b) + \theta_1 f^{abc} (D^{cd}_\mu \eta^d)\eta^b \Big)$$

$$= -\frac{\theta_2}{g}\Big(D^{ab}_\mu (\delta_{\theta_1}\eta^b) + \underbrace{\theta_1 f^{abc} (\partial_\mu \eta^c)\eta^b}_{2^{nd}\text{ term.}} - g\theta_1 \underbrace{f^{abc} f^{cde} A^e_\mu \eta^d \eta^b}_{3^{rd}\text{ term.}} \Big)$$

→ for the second term: use the fact that f^{abc} is antisymmetric & replace $(\partial_\mu \eta^c)\eta^b$ with its antisymmetric part!

$$(\partial_\mu \eta^{[c})\eta^{b]} = \frac{1}{2}(\partial_\mu \eta^c)\eta^b - \frac{1}{2}(\partial_\mu \eta^b)\eta^c$$

$$= \frac{1}{2}(\partial_\mu \eta^c)\eta^b + \frac{1}{2}\eta^c(\partial_\mu \eta^b)$$ (η: Grassmann; moved to the front.) ↓ pick (−) sign.

$$= \frac{1}{2}\partial_\mu(\eta^c\eta^b)$$

→ for the third term: use the antisymmetry of $\eta^d\eta^b$ to replace $f^{abc}f^{cde}$ with its antisym. part:

$$\frac{1}{2}\left(f^{abc}f^{cde} - f^{adc}f^{cbe}\right) = -\frac{1}{2}\left[(T^b_A)^{ac}(T^d_A)^{ce} - (T^d_A)^{ac}(T^b_A)^{ce}\right]$$

$$= -\frac{1}{2} i f^{bdh} (T^h_A)^{ae}$$

$$\underbrace{= -\frac{1}{2} f^{bdh} f^{hae}}_{\text{Jacobi identity.}}$$

↳ Now we have:

$$= -\frac{\theta_2}{g}\Big(D^{ab}_\mu (\delta_{\theta_1}\eta^b) + \frac{\theta_1}{2} f^{abc}\partial_\mu(\eta^c\eta^b) + g\frac{\theta_1}{2} f^{bdh} f^{hae} A^e_\mu \eta^d \eta^b$$

$b \to h$

$$= -\frac{\theta_2}{g}\Big(\underbrace{D^{ah}_\mu (\delta_{\theta_1}\eta^h)}_{b\to h} + \frac{\theta_1}{2}\big(\delta^{ah}\partial_\mu - g \underbrace{f^{ahe}}_{= -f^{hae}} A^e_\mu\big) \underset{f^{bca}}{f^{bch}} \eta^c \eta^b$$

1st term: $\underbrace{\delta^{ah} f^{bch}}_{f^{bca}}$

2nd term: $d \to c$

$$= -\frac{\theta_2}{g} D^{ah}_\mu \Big(\underbrace{\delta_{\theta_1}\eta^h}_{\frac{\theta_1}{2} f^{hcb}\eta^c\eta^b\ \text{(given.)}} + \frac{\theta_1}{2} f^{bch} \eta^c \eta^b \Big)$$ → vanishes since $f^{hcb} = -f^{bch}$.

(BRST Jacobian).

EX2 We will proceed like in the alternative solution of a).
This means we need to calculate the Jacobian matrix for the simultaneous BRST-transformation of all the fields $\{A_\mu^a, \psi_i, \bar{\psi}_i, \eta^a, \bar{\eta}^a\}$. Let's list the transformations:

- $A'^a_\mu = A_\mu^a - \frac{\theta}{g} D_\mu^{ab} \eta^b = A_\mu^a - \frac{\theta}{g}\left(\partial_\mu \eta^a - g f^{abc} \eta^b A_\mu^c \right)$
- $\eta'^a = \eta^a + \frac{\theta}{2} f^{abc} \eta^b \eta^c$
- $\bar{\eta}'^a = \bar{\eta}^a + \frac{1}{g} \theta w^a$
- $\psi'_i = \psi_i - i T^a_{ij} \theta \eta^a \psi_j$
- $\bar{\psi}'_i = \bar{\psi}_i + i \bar{\psi}_j T^a_{ji} \theta \eta^a$

→ Non-vanishing elements of the Jacobian matrix.

- $\frac{\partial A'^a_\mu}{\partial A_\nu^b} = \delta_\mu^\nu \left(\delta^{ab} - \theta f^{abc} \eta^c \right)$
- $\frac{\partial A'^a_\mu}{\partial \eta^b} = \overset{\{\eta,\theta\}=0}{-} \theta f^{abc} A_\mu^c$
- $\frac{\partial \eta'^a}{\partial \eta^b} = \delta^{ab} - \frac{\theta}{2} \frac{\partial}{\partial \eta^b}\left(f^{acd} \eta^c \eta^d \right) = \delta^{ab} - \frac{\theta}{2}\left(f^{abc}\eta^c - f^{acb}\eta^c \right) = \delta^{ab} - \theta f^{abc} \eta^c$
- $\frac{\partial \bar{\eta}'^a}{\partial \bar{\eta}^b} = \delta^{ab}$, $\frac{\partial \psi'_i}{\partial \psi_j} = \delta_{ij} - i T^a_{ij} \theta \eta^a$, $\frac{\partial \psi'_i}{\partial \eta^a} = i T^a_{ij} \theta \psi_j$,
- $\frac{\partial \bar{\psi}'_i}{\partial \bar{\psi}_j} = \delta_{ij} + i T^a_{ji} \theta \eta^a$, $\frac{\partial \bar{\psi}'_i}{\partial \eta^a} = i \bar{\psi}_j T^a_{ji} \theta$

all other combinations are zero.

b) (cont.) We take the effort of writing out the whole Jacobian matrix.

$$J = \left(\begin{array}{c|cccc} \frac{\partial A'^a_\mu}{\partial A^b_\nu} & \frac{\partial A'^a_\mu}{\partial \eta^b} & \frac{\partial A'^a_\mu}{\partial \bar\eta^b} & \frac{\partial A'^a_\mu}{\partial \psi_j} & \frac{\partial A'^a_\mu}{\partial \bar\psi_j} \\ \hline \frac{\partial \eta'^a}{\partial A^b_\mu} & \frac{\partial \eta'^a}{\partial \eta^b} & \frac{\partial \eta'^a}{\partial \bar\eta^b} & \frac{\partial \eta'^a}{\partial \psi_j} & \frac{\partial \eta'^a}{\partial \bar\psi_j} \\ \frac{\partial \bar\eta'^a}{\partial A^b_\mu} & \frac{\partial \bar\eta'^a}{\partial \eta^b} & \frac{\partial \bar\eta'^a}{\partial \bar\eta^b} & \frac{\partial \bar\eta'^a}{\partial \psi_j} & \frac{\partial \bar\eta'^a}{\partial \bar\psi_j} \\ \frac{\partial \psi'_i}{\partial A^b_\mu} & \frac{\partial \psi'_i}{\partial \eta^b} & \frac{\partial \psi'_i}{\partial \bar\eta^b} & \frac{\partial \psi'_i}{\partial \psi_j} & \frac{\partial \psi'_i}{\partial \bar\psi_j} \\ \frac{\partial \bar\psi'_i}{\partial A^b_\mu} & \frac{\partial \bar\psi'_i}{\partial \eta^b} & \frac{\partial \bar\psi'_i}{\partial \bar\eta^b} & \frac{\partial \bar\psi'_i}{\partial \psi_j} & \frac{\partial \bar\psi'_i}{\partial \bar\psi_j} \end{array}\right) \equiv \begin{pmatrix} A & D \\ C & B \end{pmatrix}$$

where A, B have commuting entries and C, D anticommuting ones. As in Ex. 5.1.

$\rightarrow$ since $\boxed{C = 0}$, the formula for the superdeterminant is just

$$\boxed{\operatorname{sdet} J = \frac{\det A}{\det B}}$$

Looking at the matrix-entries we computed on p. 5, we see that both A and B are of the structure $\boxed{\mathbb{1} + \theta M}$, with M being a Grassmannian matrix.

$$\Rightarrow \det(\mathbb{1} + \theta M) = \det \mathbb{1} + \theta \frac{d}{d\theta} \det(\mathbb{1} + \theta M)\Big|_{\theta = 0} = \det(\mathbb{1}) + \theta \operatorname{tr}(M)$$

[For invertible matrices: $d \det(M) = \det(M) \operatorname{tr}(M^{-1} dM)$]

$$\Rightarrow \operatorname{sdet} J = \frac{1 + \theta \operatorname{tr}(M_A)}{1 + \theta \operatorname{tr}(M_B)} = 1 + \theta\left(\operatorname{tr}(M_A) - \operatorname{tr}(M_B)\right)$$

$$= 1 + \theta\Big[\underbrace{\operatorname{tr}\left(\overset{\delta_\mu^\mu}{4} f^{abc} \eta^c\right)}_{=0 \text{ (f antisymm.)}} - \operatorname{tr}\left(-f^{abc} \eta^c - i T^a_{ij} \eta^a + i T^a_{ji} \eta^a\right) =$$

$$= 1 + \theta\Big[\underbrace{-\operatorname{tr}\left(f^{abc} \eta^c\right)}_{=0} \underbrace{- i \operatorname{tr}\left(T^a_{ij} \eta^a\right) + i \operatorname{tr}\left(T^a_{ji} \eta^a\right)}_{=0}\Big] = \underline{1}$$

The Jacobian of the transformation is 1, i.e. the BRST-transformations leave the path integral measure invariant.

Sheet 7

$\boxed{\text{Ex3}}$ (a) When the gauge group is Abelian $\Rightarrow f^{abc}=0$ & $D_\mu^{ab} \to \partial_\mu \delta^{ab}$

hence the EoM become: $\begin{cases} \partial^2 \eta^a = 0 \\ \partial^2 \bar{\eta}^a = 0 \\ \partial^2 w^a = 0 \end{cases}$

So $\eta, \bar{\eta}, w$ completely decouple & satisfy free field EoM.

(b) $Q_B = \int d^3x \left[w^a D_0^{ab} \eta^b - \dot{w}^a \eta^a + \frac{1}{2} i g \dot{\bar{\eta}}^a f^{abc} \eta^b \eta^c \right]$

$\Downarrow$ when gauge group is Abelian

$Q_B = :\int d^3x \left[w^a \dot{\eta}^a - \dot{w}^a \eta^a \right]:$

It makes sense to apply normal ordering now to this expression because one is now dealing with free fields which one can mode expand as follows:

$$\begin{cases} w = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[a_w e^{-ip\cdot x} + a_w^\dagger e^{ip\cdot x} \right] \\ \eta = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[a_\eta e^{-ip\cdot x} + a_\eta^\dagger e^{ip\cdot x} \right] \end{cases}$$

Note: One can drop the label "a" since now $a=1$ for Abelian group.

Have: $\begin{cases} :\int d^3x\, w^a \dot{\eta}^a: = \int d^3x \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \int \frac{d^3q}{\sqrt{(2\pi)^3 2E_q}} : \left[a_w e^{-ip\cdot x} + a_w^\dagger e^{ip\cdot x} \right] \left[-iE_q a_\eta e^{-iq\cdot x} + iE_q a_\eta^\dagger e^{iq\cdot x} \right]: \\ :\int d^3x\, \dot{w}\eta: = \int_x \int_p \int_q : \left[-iE_p a_w e^{-ip\cdot x} + iE_p a_w^\dagger e^{ip\cdot x} \right] \left[a_\eta e^{-iq\cdot x} + a_\eta^\dagger e^{iq\cdot x} \right]: \end{cases}$

and use the fact that $[a_w, a_\eta^\dagger] = 0$ to normal order.

$\rightsquigarrow$ Doing so one obtains: $Q_B = i \int d^3p \left[a_\eta^\dagger(p) a_w(p) - a_w^\dagger(p) a_\eta(p) \right]$

(c) Using part (a) we see that the fields $\eta, \bar{\eta}$ & W satisfy free EoM & hence decouple from one another

$\Rightarrow$ the full state space $\mathcal{V}$ splits: $\mathcal{V} = \mathcal{V}_{phys} \otimes \mathcal{V}_{FP}$

where $\mathcal{V}_{phys}$ is the physical state space & $\mathcal{V}_{FP}$ contains the ghosts / anti-ghosts.

(d) Since: $|phys\rangle \equiv |f\rangle \otimes |0\rangle_{FP}$, where $|0\rangle_{FP} \in \mathcal{V}_{FP}$ is the ghost/anti-ghost vacuum & $|f\rangle \in \mathcal{V}_{phys}$.

$\rightarrow$ With this explicit form of a physical state one can determine an explicit expression for the subsidiary condition $Q_B|phys\rangle = 0$ in the Abelian theory.

$$Q_B|phys\rangle = i\int d^3p\, [a^\dagger_\eta a_W - a^\dagger_W a_\eta](|f\rangle \otimes |0\rangle_{FP})$$

$$= i\int d^3p \left\{ a_W|f\rangle \otimes a^\dagger_\eta|0\rangle_{FP} - a^\dagger_W|f\rangle \otimes \underbrace{a_\eta|0\rangle_{FP}}_{=0} \right\} \overset{!}{=} 0$$

$=0$ since lowering operator acting on vacuum by def $=0$.

As: $\int_p a^\dagger_\eta(p)|0\rangle_{FP}$ is non-zero in general, then this condition is satisfied only if $a_W(p)|f\rangle = 0 \;\forall p$, which is equivalent to the condition $W^{(+)}(x)|f\rangle = 0 \;\forall x,\; \forall |f\rangle \in \mathcal{V}_{phys}$.
where $(+)$ = +ve freq component of field $W(x)$.

- But W satisfies the EoM: $\partial^\mu A_\mu + \xi W = 0$, so the above condition is equivalent to $\boxed{(\partial_\mu A^\mu)^{(+)}|phys\rangle = 0}$ which is the "Gupta-Bleuler condition", as expected!