**Exercise 1.** ***Symmetry generators***

Consider a scalar field theory and suppose we have a symmetry $S$ of the classical action (for simplicity, let's assume it is also a symmetry of the Lagrangian) and the path integral measure, that acts linearly on the fields,

$$\phi_n(x) \to \phi_n(x) + \epsilon\delta\phi_n(x) = \phi_n(x) + i\epsilon\, t_{nm}\, \phi_m(x) \tag{1}$$

Show that the *Noether charge* $Q$ associated to this symmetry generates the symmetry transformations, i.e.

$$[Q, \phi_n(x)] = -t_{nm}\phi_m(x) \tag{2}$$

*Hint.* $Q = \int d^3x\, J^0(\vec{x}, 0)$, *with* $J^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m(x))}\delta\phi_m(x)$. *Use the canonical commutation relations for the fields and their conjugated momenta.*

**Exercise 2.** ***Example of Goldstones theorem***

Consider the following Lagrangian for $N$ real scalar fields,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\Phi)^T(\partial^\mu\Phi) - \frac{1}{2}m^2\Phi^T\Phi - \frac{\lambda}{4}\left(\Phi^T\Phi\right)^2, \tag{3}$$

where $\Phi^T = (\phi_1, \cdots, \phi_N)$. Convince yourself that it has a global $O(N)$ symmetry.

a) Now assume that $m^2 < 0$. Find the vacua by finding the minima of the effective potential $V(\Phi)$ at tree level (i.e. just the classical potential). You should find that a vacuum $\Phi_0$ satisfies

$$\Phi_0^T\Phi_0 = -\frac{m^2}{\lambda} \tag{4}$$

b) Show that the mass matrix $M$ defined by

$$M_{nm} = \left.\frac{\partial^2 V(\Phi)}{\partial\phi_n\partial\phi_m}\right|_{\Phi=\Phi_0} \tag{5}$$

has only one eigenvalue different from zero, i.e. there are $(N-1)$ massless particles in the spectrum. Why exactly $(N-1)$?

*Hints. Find an explicit eigenvector with an eigenvalue different from zero. Then use the fact that eigenvectors of symmetric matrices are orthogonal.*

**Exercise 3.** ***Feynman Parametrization of Loop Integrals***

Consider a generic scalar 2-loop integral in $d$ dimension with $n$ massless propagators $1/A_i$ raised to some powers $\nu_i$

$$J^d(\{\nu_i\}, \{Q_i^2\}) = \int \frac{d^dk_1}{i\,\pi^{d/2}} \int \frac{d^dk_2}{i\,\pi^{d/2}} \frac{1}{A_1^{\nu_1}\ldots A_n^{\nu_n}} \tag{6}$$

You already know that we can rewrite the product of the propagators using the Feynman parameters, i.e.

$$\frac{1}{A_1^{\nu_1}\dots A_n^{\nu_n}} = \frac{\Gamma(\nu_1+\dots+\nu_n)}{\Gamma(\nu_1)\dots\Gamma(\nu_n)} \int_0^1 dx_1\dots dx_n\, \delta\left(1-\sum_i x_i\right) \frac{x_1^{\nu_1-1}\dots x_n^{\nu_n-1}}{\left[\sum_i x_i A_i\right]^N} \tag{7}$$

where $N=\sum_i \nu_i$. Consider now the sum present in the denominator $\sum_i x_i A_i$. The most general form of this sum is given by

$$\sum_i x_i A_i = a\,k_1^2 + b\,k_2^2 + 2\,c\,k_1\cdot k_2 + 2\,d\cdot k_1 + 2\,e\cdot k_2 + f \tag{8}$$

where $a$, $b$, $c$, $d^\mu$, $e^\mu$, $f$ are linear in the parameters $x_i$.

(a) Performing the change of variables

$$k_1^\mu = K_1^\mu - \frac{c\,K_2^\mu}{a} + X^\mu \tag{9}$$

$$k_2^\mu = K_2^\mu + Y^\mu \tag{10}$$

where

$$X^\mu = \frac{c\,e^\mu - b\,d^\mu}{P} \tag{11}$$

$$Y^\mu = \frac{c\,d^\mu - a\,e^\mu}{P} \tag{12}$$

$$P = ab - c^2 \tag{13}$$

show that the previous sum can be rewritten as

$$\sum_i x_i A_i = aK_1^2 + \frac{P}{a}\,K_2^2 + \frac{Q}{P} \qquad Q = -a\,e^2 - b\,d^2 + 2\,c\,e\cdot d + f\,P \tag{14}$$

(b) Integrating out the shifted loop-momenta, show that the final result is

$$J^d(\{\nu_i\},\{Q_i^2\}) = (-1)^d \frac{\Gamma(N-d)}{\Gamma(\nu_1)\dots\Gamma(\nu_n)} \int_0^1 \left(\prod_i dx_i\, x_i^{\nu_i-1}\right) \delta\left(1-\sum_i x_i\right) P^{N-\frac{3d}{2}}\, Q^{d-N} \tag{15}$$

*Hint. Use the formula*

$$\int \frac{d^d k_i}{i\,\pi^{d/2}} \frac{1}{[k_i^2-\Delta]^n} = (-1)^n \frac{\Gamma(n-d/2)}{\Gamma(n)} \Delta^{d/2-n}$$

(c) For 1-loop integral we can find a similar formula. Starting from (8), show that in this case

$$P = \sum_i x_i = 1 \qquad Q = f - d^2$$

*Hint. First take the limit $c\to 0$ to eliminate the common propagator of the two loops; then take $b=0$, $e^\mu = 0$ to eliminate the propagator of the second loop*

(d) Show that the final form of the integral is now given by

$$J^d(\{\nu_i\},\,\{Q_i^2\}) = (-1)^{d/2}\,\frac{\Gamma(N-d/2)}{\Gamma(\nu_1)\ldots\Gamma(\nu_n)}\int_0^1\left(\prod_i dx_i\,x_i^{\nu_i-1}\right)\delta\left(1-\sum_i x_i\right)\,P^{N-d}\,Q^{d/2-N} \tag{16}$$

From this result we notice that the UV divergencies present in the integral, which are related to the superficial degree of divergence, are encoded in the Gamma function in the front.

(e) To convince yourself of this result, consider the 1-loop scalar triangle with

$$p_1^2 = p_2^2 = 0, \qquad (p_1+p_2)^2 = s$$

Show first that in general

$$d^\mu = \sum_i x_i\,q_i^\mu, \qquad f = \sum_i x_i\,q_i^2 \tag{17}$$

$$\text{with}\quad q_i^\mu = q_{i-1}^\mu + p_{i-1}^\mu, \qquad q_1^\mu = 0 \tag{18}$$

so that

$$Q = \sum_{j=2}^{n}\sum_{i<j} x_i\,x_j\,s_{ij}, \qquad s_{ij} = (q_i-q_j)^2 = \left(\sum_{m=i}^{j-1} p_m^\mu\right)^2$$

and $k$ is the loop momentum. Then find the final result as in (16).