Gravity waves

Free surface of a liquid in equilibrium in a gravitational field is a plane. If this surface is perturbed then the motion will occur — gravity waves.

We shall consider gravity waves in which velocity of the fluid particles is small, such that one can neglect $(\vec{v}\cdot\vec{\nabla})\vec{v}$ term compared to $\frac{\partial v}{\partial t}$ in Euler's equation. To get an estimate:

If the amplitude of the wave is a, then fluid velocity is $v \sim a/\tau$, where τ is period of the wave. It changes over distances $\sim \lambda$ (wavelength). Thus $(\bar{v}\cdot\bar{\nabla})v \sim \frac{v^2}{\lambda} \sim \frac{v a}{\lambda \tau}$

$$\frac{\partial v}{\partial t} \sim \frac{v}{\tau} \Rightarrow (\bar{v}\cdot\bar{\nabla})\bar{v} \ll \frac{\partial v}{\partial t} \text{ if } \underline{a \ll \lambda}$$

The amplitude should be much smaller than the wavelength.

We also neglect viscosity for the moment

For potential flow of an incompressible liquid $\operatorname{div} v = 0$, $\operatorname{rot} v = 0$ =>

$$v = \operatorname{grad} \varphi \quad , \quad \nabla^2 \varphi = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \underbrace{(\vec{v} \cdot \vec{\nabla}) \vec{v}}_{0} = - \frac{\vec{\nabla} p}{\rho} + \vec{g} \quad =>$$

$$p = - \rho \left(\frac{\partial \varphi}{\partial t} + g z \right)$$

The surface elevation is called $\xi(x, y)$

Then at the surface we have

$$p_0 = - \rho \left(g \xi + \frac{\partial \varphi}{\partial t} \right)$$

Atmospheric pressure p_0 can be included by redefining $\varphi \to \varphi + p_0 t / \rho$

Then $$g \xi + \left. \frac{\partial \varphi}{\partial t} \right|_{z = \xi} = 0$$

Vertical velocity at the surface $v_z = \frac{\partial \xi}{\partial t}$

But for potential flow $v_z = \frac{\partial \varphi}{\partial z}$, thus

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=\xi} = \frac{\partial \xi}{\partial t} = -\frac{1}{g}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|_{z=\xi}$$

Since the oscillations are small

$z = \xi \Leftrightarrow z = 0$ and we have finally

$$\Delta \varphi = 0$$

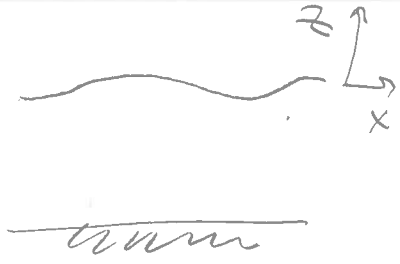
$$\left(\frac{\partial \varphi}{\partial z} + \frac{1}{g}\frac{\partial^2 \varphi}{\partial t^2}\right)_{z=0} = 0$$

One need to supplement these equations by the boundary condition at the bottom $\frac{\partial \varphi}{\partial z} = 0$ at $z = -h$.

Deep water (short waves $\lambda \ll h$)

Consider gravity wave propagating along x. Let us look for solution in form

$$\varphi = f(z)\cos(kx - \omega t)$$



Then $\nabla^2 \varphi = 0 \Rightarrow$

$$\frac{d^2 f}{dz^2} = k^2 f \Rightarrow f(z) = e^{kz}$$

Boundary condition at the surface gives us

$$\left.\left(\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial z}\right)\right|_{z=0} = 0 \Rightarrow \omega^2 = gk$$

This is dispersion relation.

For gravity waves it is nonlinear $\omega = g\sqrt{|k|}$

$$\varphi = A e^{kz}\cos(kx - \omega t)$$

The velocity distribution is obtained from $v = \nabla \varphi$

$$v_x = -A k e^{kz} \sin(kx - \omega t)$$

$$v_z = A k e^{kz} \cos(kx - \omega t)$$

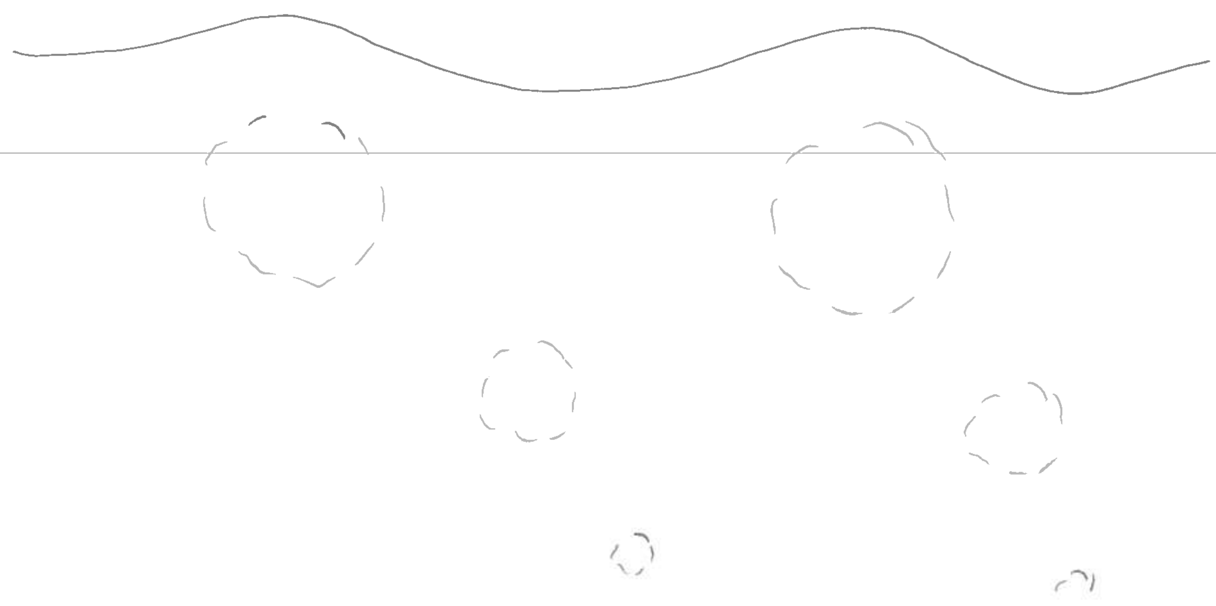
Velocity diminishes exponentially but rotates

The trajectories of fluid particles are obtained by integrating $\dot{r} = v$ near some $r_0 = (x_0, z_0)$

$$x = x_0 - A \frac{k}{\omega} e^{k z_0} \cos(k x_0 - \omega t)$$

$$z = z_0 - \frac{A k}{\omega} e^{k z_0} \sin(k x_0 - \omega t)$$

Trajectories – circles



Shallow water (long waves $\lambda >> h$)

Solving $\Delta \varphi = 0$

$$\left(g \frac{\partial \varphi}{\partial z} + \frac{\partial^2 \varphi}{\partial t^2} \right)_{z=0} = 0$$

with the boundary condition at the bottom $v_z = \left.\frac{\partial \varphi}{\partial z}\right|_{z=-h} = 0$

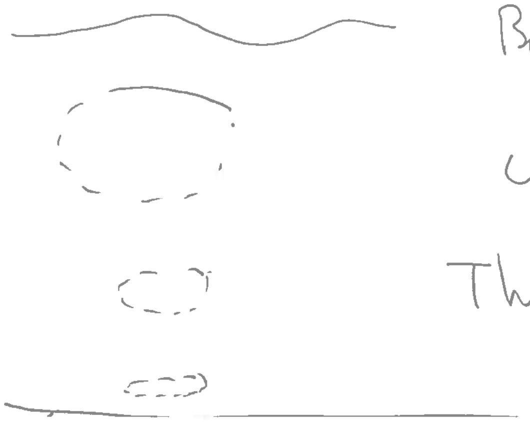
we have $\varphi = f(z) \cos(kx - \omega t)$

$$\frac{d^2 f}{dz^2} = k^2 f$$

$$f'(z=-h) = 0 \Rightarrow f = \cosh(k(z+h)) \Rightarrow$$

$$\varphi(x,z,t) = A \cosh[k(z+h)] \cos(kx - \omega t)$$

Trajectories - ellipses



Boundary condition at the top

$$\omega^2 = kg \tanh(kh)$$

Thus for $kh << 1$ linear dispersion

$$\omega = \sqrt{gh}\, k$$

Viscosity - damping of gravity waves

Consider small viscosity $\lambda^2 \omega \gg \nu$

We still can neglect $(\vec{v}\cdot\vec{\nabla})\vec{v}$ term in the Navier-Stokes equation. Then our solution of the potential flow would be good approximation. But it doesn't satisfy boundary conditions at the top (since $\sigma_{xz} = -2\eta \varphi_{xz}$ and $\sigma'_{zz} = -2\eta \varphi_{zz} \neq 0$) and at the bottom $v_{\parallel} = \frac{\partial \varphi}{\partial x} \neq 0 \Rightarrow$ boundary layers

For small viscosity these layers at the top and at the bottom are thin and can be neglected (we assume deep water $kh \gg 1$)

The change in energy per unit time is

$$\frac{dE}{dt} = -\frac{\eta}{2} \int \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)^2 dV$$

Introducing potential $\frac{\partial v_i}{\partial x_k} = \frac{\partial^2 \varphi}{\partial x_i \partial x_k} = \frac{\partial v_k}{\partial x_i}$

$$\frac{dE}{dt} = -2\eta \int \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_k}\right)^2 dV =$$

$$= -2\eta \int \left(\varphi_{xx}^2 + \varphi_{zz}^2 + 2\varphi_{xz}^2\right) dV$$

The potential $\varphi = \varphi_0 \cos(kx - \omega t + \alpha)\, e^{kz} \Rightarrow$

averaged over period

$$\overline{\frac{dE}{dt}} = \int_0^{2\pi/\omega} \frac{dE}{dt}\, dt\, \frac{\omega}{2\pi} = -8\eta k^4 \int \overline{\varphi^2}\, dV$$

The average energy is from the virial theorem

$$\overline{E} = \int \rho \overline{v^2}\, dV = \rho \int \overline{\left(\frac{\partial \varphi}{\partial x_i}\right)^2}\, dV = 2\rho k^2 \int \overline{\varphi^2} dV$$

Thus the damping coefficient

$$\gamma = \frac{\overline{\dot{E}}}{2E} = 2\nu k^2 = 2\nu \frac{\omega^4}{g^2}$$

Then $E \propto e^{-2\gamma t}$, amplitude of the wave $\propto e^{-\gamma t}$

Dimensional estimates

Dispersion laws that we derived can be guessed from dimensional reasoning. Really for deep water the only parameters that enter are ω, k, and g

$$[\omega] = \frac{1}{sec} \quad [g] = \frac{m}{sec^2}, \quad [k] = \frac{1}{m} \Rightarrow$$

$$\omega^2 \propto gk$$

Analogously for shallow water one can make [velocity] out of g and h, $c \propto \sqrt{gh}$

$$\omega \propto \sqrt{gh}\, k$$

Note that the wave speed $c = \sqrt{gh}$ in the relation $\omega = ck$ is equal to the falling speed from the height h. Also

$$\omega^2 = gk \iff r = gt^2$$

thus circles after the stone has been droped in the water expand with the acceleration of the free fall!

Circular waves on the deep water

We are looking for solution in the form

$$\varphi \propto e^{kz} e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

Then $\nabla^2\varphi = 0$ and from the boundary condition $g\frac{\partial\varphi}{\partial z} + \frac{\partial^2\varphi}{\partial t^2} = 0 \Rightarrow \omega^2 = gk$

Initially perturbation was at $r \approx 0 \Rightarrow$

$$\varphi(z=0) \propto \int e^{i\vec{k}\cdot\vec{r} - i\omega t} d^2k \quad \text{with } \omega^2 = gk$$

$$v \propto \int e^{i\vec{k}\cdot\vec{r} - i\sqrt{gk}\,t}$$

To calculate such integrals for large t limit one uses <u>method of stationary phase</u>

Consider the following integral

$$f(t) = \int_{x_1}^{x_2} dx \; e^{ith(x)} \qquad \text{in the limit } t \to \infty$$

1) If $h'(x) \neq 0$ for $x_1 \leq x \leq x_2$ then

$$f(t) = \frac{1}{it} \int_{x_1}^{x_2} \frac{dx}{h'(x)} \left(e^{ith(x)} \right)' =$$

$$= \frac{1}{it} \frac{e^{ith(x)}}{h'(x)} \Bigg|_{x_1}^{x_2} + \frac{1}{it} \int_{x_1}^{x_2} \frac{dx}{h''(x)} e^{ith(x)} = O(t^{-1})$$

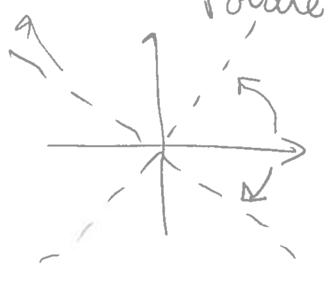
2) If $h'(x_0) = 0$, $h''(x_0) \neq 0$ then

$$f(t) = \int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \; e^{ith(x)} + O(t^{-1})$$

Expanding close to x_0, $h(x) = h(x_0) + h''(x_0) \frac{(x-x_0)^2}{2} + \dots$

$$f(t) = e^{ith(x_0)} \int_{-\varepsilon}^{+\varepsilon} ds \; e^{\frac{it s^2 h''(x_0)}{2}} =$$

$$\simeq \frac{e^{ith(x_0)}}{\sqrt{t|h''(x_0)|}} \int_{-\infty}^{\infty} d\tau \; e^{i \frac{\mathrm{sgn}\, h''(x_0) \tau^2}{2}} = \text{rotate}$$



$$= \frac{e^{ith(x_0)} \sqrt{2\pi} \; e^{i\frac{\pi}{4} \mathrm{sgn}[h''(x_0)]}}{\sqrt{t|h''(x_0)|}} \qquad \propto \frac{1}{\sqrt{t}}$$

Applying it to the integral

$$v \propto \int e^{i\vec{k}\vec{r} - i\sqrt{g|k|}\,t} (d^2k) =$$

$$= \int e^{ikr\cos\varphi - i\sqrt{gk}\,t}\, k\,dk\,d\varphi$$

extremum of the exponent is at

$$\frac{r\cos\varphi}{t} = \frac{\sqrt{g}}{2\sqrt{k}}, \Rightarrow k_0 = \frac{gt^2}{4r^2\cos^2\varphi}$$

$$h = \frac{kr\cos\varphi}{t} - \sqrt{gk} \qquad h(k_0)t = \frac{gt^2}{4r\cos\varphi}$$

$$h''(k_0) = \frac{g^{1/2}}{4\,k_0^{3/2}} \propto \frac{r^3\cos^3\varphi}{gt^3} \Rightarrow$$

$$v \propto \frac{t^3 g^{3/2}}{r^{7/2}} \int \frac{d\varphi\; e^{-i\frac{gt^2}{4r\cos\varphi}}}{(\cos\varphi)^{7/2}}$$

Performing saddle point for angular integration $\left(\cos\varphi \propto 1 - \frac{\varphi^2}{2}\right)$ we obtain

$$v \propto \frac{gt^2}{r^3}\cos\frac{gt^2}{4r} \xRightarrow{n\text{-th}} \text{maximum moves with } r_n = \frac{gt^2}{8\pi n}$$