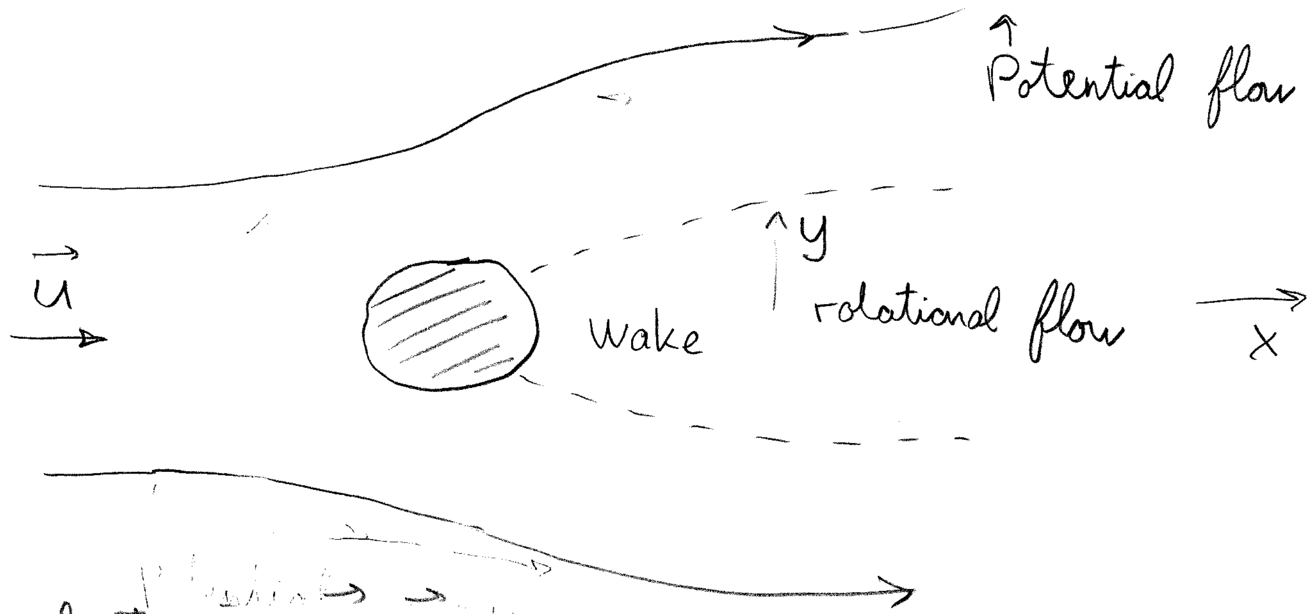
The laminar wake



- velocity is $\vec{u}+\vec{v}$

The wake is due to fluid particles that move along the streamlines passing close to the body. Away from the wake the flow at large distances is potential.

- The wake is relatively narrow. Indeed in the Navier-Stokes equation $\frac{\partial^2 v}{\partial x^2} << \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 v}{\partial z^2}$

On the other hand $(\vec{v}\cdot\vec{\nabla})v \simeq u \frac{\partial v}{\partial x}$

If at the distance x away from the body the width of the wake is Y then

$$(v\cdot\nabla)v \simeq u\frac{\partial v}{\partial x} \sim \frac{uv}{x}$$

$$\nu\nabla^2 v \sim \nu\frac{\partial^2 v}{\partial y^2} \sim \frac{\nu v}{y^2}$$

Comparing them we obtain

$$y \sim \sqrt{\frac{\nu x}{u}} \sim x\sqrt{\frac{\nu}{ux}} << x,$$

since we consider region far from the body $\frac{xu}{\nu} >> 1$

Note that the pressure doesn't change much across the wake.

$$\vec{\nabla} p = -\rho(\vec{v}\vec{\nabla})\vec{v} - \eta\Delta\vec{v}$$

Since v mainly has v_x component so does grad p $\Rightarrow \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} << \frac{\partial p}{\partial x}$

Inside the wake we can approximate the Navier-Stokes equation as

$$u\frac{\partial v_x}{\partial x} = \nu\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right)v_x$$

This is diffusion like equation.
One solves it going to the Fourier space

$$v_x(k,x) = \int v_x(r)\, e^{i(k_y y + k_z z)}\, dy\, dz$$

$$u \frac{\partial v_x(k)}{\partial x} = -\nu k^2 v_x \quad \Rightarrow$$

$$v_x(k,x) \propto e^{-\frac{\nu k^2}{u} x}$$

- $$v_x \propto \int e^{-\frac{\nu k^2 x}{u} - i k_y y - i k_z z} \frac{dk_y\, dk_z}{(2\pi)^2}$$

$$v_x \propto \frac{1}{\nu x} \exp\left[-\frac{u(z^2+y^2)}{4\nu x}\right]$$

- Thus, indeed transverse size

$$y \sim \sqrt{\nu x / u}$$

The Reynolds number in the wake is

$$Re \simeq \frac{v_x y}{\nu} \propto x^{-1/2} \to 0 \quad \Rightarrow$$

the distant wake is laminar.

Drag with a wake. Solution

The total momentum flux transported by the fluid through any closed surface is equal to the rate of momentum change which is equal to the force acting on the body.

$$F_i = \oint \Pi_{ik}\, dS_k = \oint \left[(p_0 + p')\delta_{ik} + \rho(u_i + v_i)(u_k + v_k)\right] dS_k$$

p_0 is pressure at infinity

$$\oint (p_0 \delta_{ik} + \rho u_i u_k)\, dS_k = \oint \text{const}\, d\vec{S} = 0$$

Mass conservation means, that

$$\rho \oint v_k\, dS_k = 0$$

Far from the body $v \ll u$, thus

$$F_i = \oint (p' \delta_{ik} + \rho u_k v_i)\, dS_k$$

Choosing surface as

we have

- $$F_x = \left(\iint_{x_0} - \iint_{X}\right)(p' + \rho u v_x)\, dy\, dz$$

Outside the wake we have potential flow and therefore Bernoulli's equation

$$p + \rho \frac{(u+v)^2}{2} = p_0 + \rho \frac{u^2}{2} \implies$$

- $$p' \approx -\rho u v_x$$

and the integral outside the wake vanishes. Inside the wake as we argued before the pressure is about the same as outside but v_x is much larger

Then we obtain the dragg formula

$$F_x = -\rho u \iint_{wake} v_x \, dy dz$$

$$v_x < 0 \Rightarrow F_x > 0$$

Note that the integral is equal to the deficit of fluid flux through the wake area.

- The wake also determines the <u>lift force</u> $\perp \vec{i}$

$$F_y = \rho u \left(\int_{x_0} - \int_{x} \right) v_y \, dy dz$$

The lift force per unit length of the wing can be related to the velocity circulation around it. Indeed, adding two (vanishing) integrals of v_x over two $y = \pm$ const lines we get

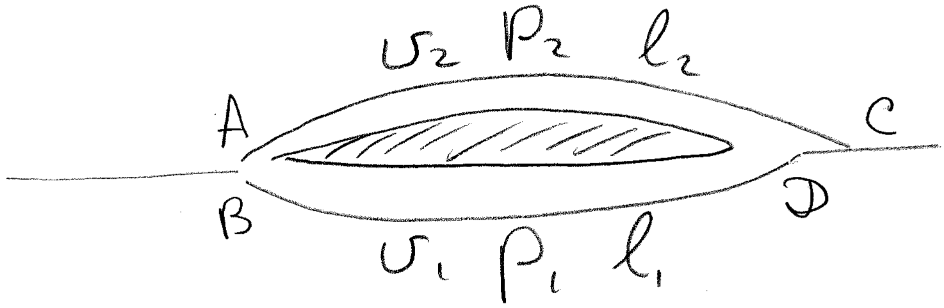
$$f_y = \rho u \left(\int_{x_0} - \int_{x} \right) v_y dy = \rho u \oint v \cdot dl$$

This is Zhukovskii's theorem

N. E. Zhukovskii (1906)

One can often here a simple explanation of the lift of the wing as being the result of

$v_2 > v_1 \Rightarrow P_2 < P_1$



This is basically true. Indeed the circulation over the closed contour ACDB is non-zero ($v_2 l_2 > v_1 l_1$). However, it would be wrong to argue that $v_2 > v_1$ because $l_2 > l_1$ – neighbouring fluid elements A, B do not meet again at the trailing edge; C is shifted relative to D.