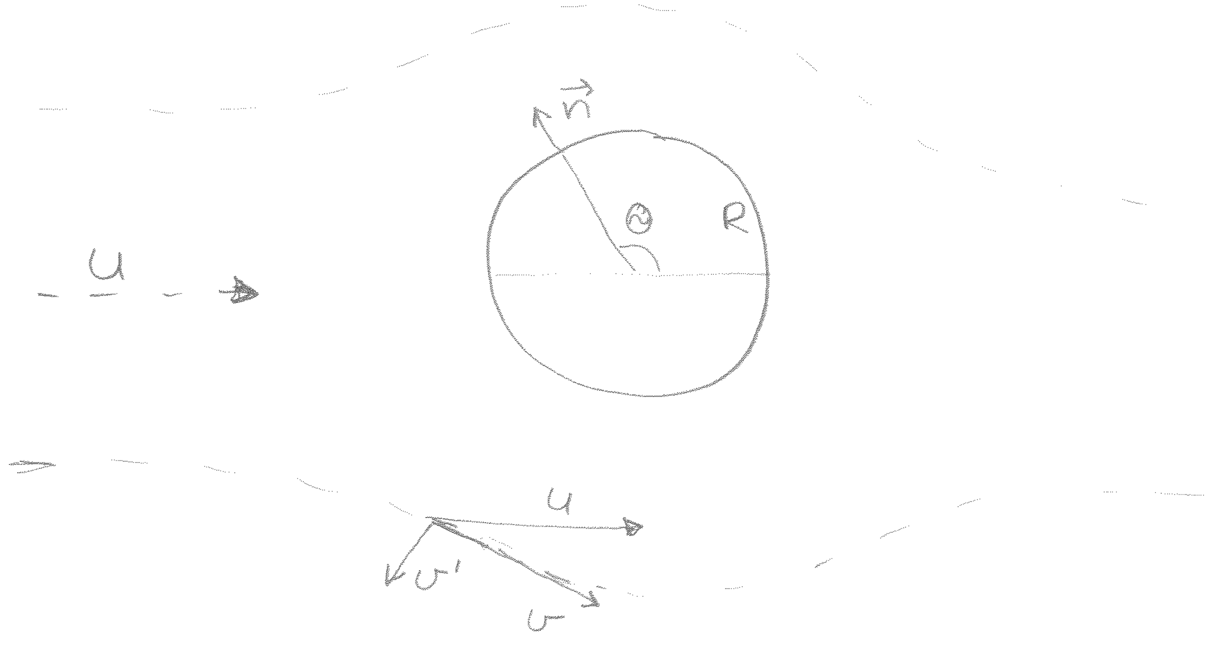
Flow with small Reynolds numbers

Stokes formula

We consider a flow of an incompressible fluid which is so viscous that the Reynolds number, $Re = \frac{uR}{\nu} << 1$. In this case the nonlinear term $(v \cdot \nabla) v$ can be neglected compared to the viscous one $\nu \nabla^2 v$ and the steady Navier-Stokes equation is

$$\eta \nabla^2 \vec{v} = \vec{\nabla} p$$

Consider motion of a sphere in a viscous fluid. We go to the reference frame of the sphere. Then fluid velocity at infinity is $\vec{u}$



Velocity of the fluid is $v = u + v'$, where $\vec{v}' \to 0$ at $r \to \infty$

For incompressible fluid $\operatorname{div} \vec{v} = 0 \Rightarrow$
$\operatorname{div} \vec{v}' = 0 \Rightarrow \vec{v}' = \operatorname{rot} \vec{A}$

The vector $\vec{A}$ must be axial, in order for its rot to be polar, like the velocity. It also has to be linear in $\vec{u}$ because the equation of motion and the boundary conditions are linear.

The only way to make an axial vector from $\vec{r}$ and $\vec{u}$ is $\vec{r} \times \vec{u} \Rightarrow \vec{A} = f'(r)\, \vec{n} \times \vec{u}$, $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$

We just reduced our problem from finding a vector field to finding a scalar function $f(r$

The vector $f'(r)\vec{n} = \operatorname{grad} f(r) \Rightarrow$

$$v' = \operatorname{rot} A = \nabla \times [\nabla f(r) \times \vec{u}]$$

Since $\vec{u} = \text{const}$ one can take ∇ out, then

$$v' = \operatorname{rot}\operatorname{rot}(f(r)\vec{u})$$

Taking rot from the equation $\eta \nabla^2 \vec{v} = \nabla p$
we obtain $\nabla^2 \operatorname{rot} \vec{v} = 0$

Expressing $\vec{v}$ via $f(r)$:

$$\operatorname{rot} \vec{v} = \nabla \times \nabla \times [\nabla f \times \vec{u}] = (\operatorname{grad}\operatorname{div} - \nabla^2)[\nabla f \times u] =$$

$$= -(\nabla^2 \nabla f) \times u$$

We used here that $\operatorname{div}[\nabla f \times u] = \operatorname{div} \operatorname{rot}(f \vec{u}) = 0$

Thus $\Delta \operatorname{rot} \vec{v} = 0 \Leftrightarrow$

$$\Delta^2 \vec{\nabla} f \times \vec{u} = 0$$

Since $\nabla f \parallel \vec{n}$ so $\Delta^2 \nabla f$ cannot be always parallel to $\vec{u}$ and we get

$$\Delta^2 \operatorname{grad} f = 0 \quad \Rightarrow$$

$$\Delta^2 f = \text{const}$$

Since at infinity v' and its derivatives must be zero $\Rightarrow$ this const $= 0 \Rightarrow$

$$\Delta^2 f = 0$$

$$\Delta^2 f = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\Delta f\right) = 0 \Rightarrow$$

$$\Delta f = \frac{2a}{r} + c$$

because $v' = v - u = 0$ at infinity $c = 0$,

then

$$f = ar + \frac{b}{r}$$

- Taking rot twice we obtain

$$\vec{v} = \vec{u} + \text{rot rot } f\vec{u} = \vec{u} + (\text{grad div} - \nabla^2)\left(ar + \frac{b}{r}\right) =$$

$$= \vec{u} - a\frac{(\vec{u} + \vec{n}(\vec{u}\cdot\vec{n}))}{r} + b\frac{(3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u})}{r^3}$$

- Boundary condition $\vec{v}(r=R) = 0 \Rightarrow$

$$\vec{u}\left(1 - \frac{a}{R} - \frac{b}{R^3}\right) + \vec{n}(\vec{u}\cdot\vec{n})\left(\frac{3b}{R^3} - \frac{a}{R}\right) = 0$$

Since this equation holds for all $\vec{n}$ then both brackets should vanish $\Rightarrow$

$$a = \frac{3b}{R^2}, \quad b = \frac{R^3}{4}, \quad a = \frac{3}{4}R$$

Thus we have finally

$$f = \frac{3}{4} R r + \frac{R^3}{4r}$$

$$\vec{v} = -\frac{3}{4} R \frac{\vec{u} + \vec{n}(\vec{u}\cdot\vec{n})}{r} - \frac{1}{4} R^3 \frac{\vec{u} - 3\vec{n}\,\vec{u}\cdot\vec{n}}{r^3} + \vec{u}$$

In spherical coordinates

$$v_r = u \cos\theta \left[1 - \frac{3R}{2r} + \frac{R^3}{2r^3} \right]$$

$$v_\theta = -u \sin\theta \left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3} \right]$$

Pressure is given by

$$\text{grad } p = \eta \Delta \vec{v} = \eta \Delta \text{rot rot}(f\vec{u}) =$$

$$= \eta \Delta (\text{grad div}(f u) - u \Delta f)$$

But $\Delta^2 f = 0 \Rightarrow$

$$\text{grad } p = \text{grad}\left[\eta \Delta \text{div}(f u)\right] = \text{grad}(\eta\, \vec{u} \cdot \text{grad}\, \Delta f) \Rightarrow$$

$$p = p_0 + \eta\, \vec{u} \cdot \text{grad}\, \Delta f = p_0 - \frac{3}{2} \eta \frac{\vec{u}\cdot\vec{n}}{r^2} R$$

Stokes formula for the drag

The force acting on a unit surface is a momentum flux through surface. On a solid surface $\vec{v} = 0$ and $F_i = -\sigma_{ik} n_k = p n_i - \sigma'_{ik} n_k$

It is clear from the symmetry of the problem that in our case force is along $\vec{u}$ =>

$$F_x = \oint \left(-p \cos\theta + \sigma'_{rr} \cos\theta - \sigma'_{r\theta} \sin\theta \right) dS$$

$$p(R) = -\frac{3}{2} \eta \frac{u \cos\theta}{R}$$

$$\sigma'_{rr} = 2\eta \frac{\partial v_r}{\partial r} = 0$$

$$\sigma'_{r\theta} = \eta \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) = -\frac{3\eta}{2R} u \sin\theta$$

Then $F_x = \dfrac{3}{2} \dfrac{\eta u}{R} \displaystyle\int dS = 6\pi \eta u R$

Stokes (1851)

Note that up to 6π factor we could guess this result from dimensional arguments.

It is clear that the law of decay $v \propto \frac{1}{r}$ cannot be realized at arbitrary large distances. To see this let us estimate $(v \cdot \nabla) v$ that was neglected.

$$(v\,\nabla)v \simeq (\vec{U}\cdot\vec{\nabla})\,\vec{v} \simeq \frac{u^2 R}{r^2}$$

Compared to the viscous term $\nu \nabla^2 v \simeq \nu \frac{uR}{r^3}$

So $(\vec{v}\cdot\vec{\nabla})\vec{v} \ll \nu \nabla^2 v$ only for

$$r \ll \frac{\nu}{u}$$

One can call $\frac{\nu}{u}$ the width of the boundary layer. The Stokes flow is realized inside the boundary layer under the assumption that the size of the body is much less than the width of boundary layer.

To find the flow at large distances of the body we have to include the term $(\vec{v}\cdot\vec{\nabla})\vec{v}$.

Since at large distances $\vec{v}$ is almost the same as $\vec{u}$ we can approximate $(\vec{v}\cdot\vec{\nabla})\vec{v}$ by $(\vec{u}\cdot\vec{\nabla})\vec{v}$. Then velocity at large distances obeys the linear equation

- $$(\vec{u}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}P}{\rho} + \nu\nabla^2\vec{v}$$

C.W. Oseen (1910)

Solving it for the sphere one can find correction to the Stokes formula

- $$F = 6\pi\eta u R\left(1+\frac{3}{8}Re\right) = 6\pi\eta u R\left(1+\frac{3uR\rho}{8\eta}\right)$$

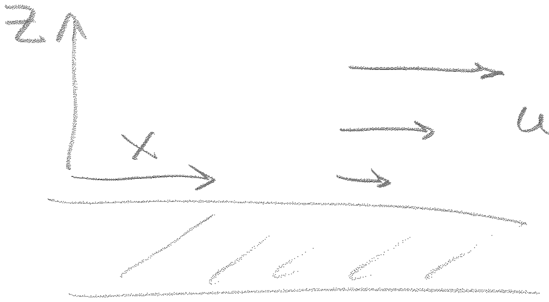
For the cylinder moving perpendicular to its axis one should use Oseen's equation from the start, then

$$F = \frac{4\pi\eta u}{\ln(3.70\,\nu/uR)}$$

What is the flow outside the boundary layer, for $r > \frac{\nu}{u}$? Is it potential? Not quite.

For finite Re, there is an infinite region (called the wake) behind the body where it is impossible to neglect viscosity at any distances.

The point is that viscosity produces vorticity in the boundary layer



$$\omega_y = \frac{dv_x}{dz}$$

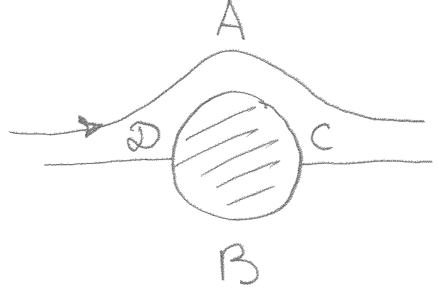
The wake is reached by the fluid particles which move along the streamlines passing close to the body. The rotation flow near the body is then transported into the wake region by convection.

Let us describe how the wake arises for $Re \gg 1$ 16

Wake appears due to phenomenon called separation.

(L. Prandtl 1905)

Consider e.g. the flow around a cylinder

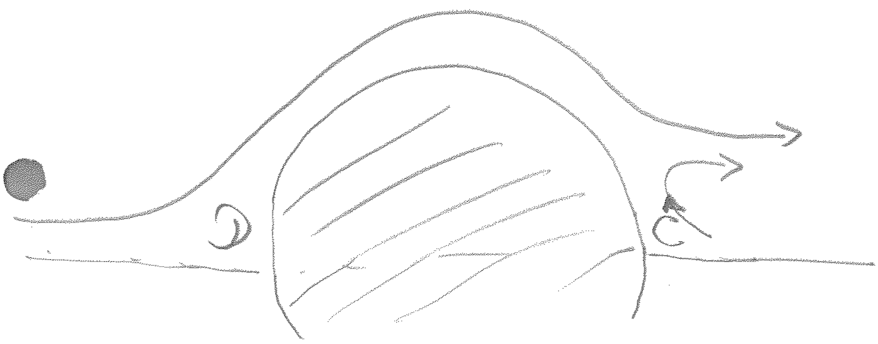


Ideal flow is symmetrical with respect to AB

On the upstream half DA the fluid accelerates, the pressure decreases according to Bernoulli law

On the downstream part AC, the reverse happens and the fluid particle move against the pressure gradient. Viscosity varies pressure only slightly across the boundary layer $\Rightarrow$ the pressure is almost equal to that of an ideal fluid flow. But the velocity of the fluid particles that reach the points A and B lower in viscous fluid because of dissipation.

Then those particles have insufficient energy to overcome the pressure gradient downstream
The particle motion is stopped by pressure gradient before tho point C is reached.



The pressure gradient then accelerates the particles from the point C upwards so that separation and recirculating vortex appear.