Hooke's law

In equilibrium $\sigma_{ik} = 0 = \frac{\partial F}{\partial u_{ik}}$ =>

Free energy F is quadratic in u_{ik}

From symmetric tensor one can make two quadratic scalars =>

$$F = F_0 + \frac{\lambda}{2} u_{ii}^2 + \mu u_{ik}^2$$

λ, μ - Lamé coefficients

If $u_{ii} = 0$ => volume is unchanged => shear

μ - shear modulus

$$u_{ik} = \underbrace{\left(u_{ik} - \frac{1}{3}\delta_{ik} u_{ll}\right)}_{\text{shear}} + \underbrace{\frac{1}{3}\delta_{ik} u_{ll}}_{\text{compression}}$$

$$F = \mu \left(u_{ik} - \frac{\delta_{ik} u_{ll}}{3}\right)^2 + \frac{K}{2} u_{ll}^2$$

K - compression modulus $K = \lambda + \frac{2}{3}\mu$

If $u_{ll} = 0$ then only the first term $F = \mu(\;)^2$

if $u_{ik} \propto \delta_{ik}$ then only the second $\frac{K u_{ll}^2}{2}$ =>

$K > 0, \mu > 0$

$$dF = \left[K u_{ll} \delta_{ik} + 2\mu \left(u_{ik} - \frac{\delta_{ik} u_{ll}}{3} \right) \right] du_{ik} \Rightarrow$$

$$\sigma_{ik} = K u_{ll} \delta_{ik} + 2\mu \left(u_{ik} - \frac{1}{3} \delta_{ik} u_{ll} \right)$$

And inverting it we obtain

$$u_{ik} = \frac{1}{9K} \delta_{ik} \sigma_{ll} + \frac{1}{2\mu} \left(\sigma_{ik} - \frac{1}{3} \delta_{ik} \sigma_{ll} \right)$$

Hooke's law

- Homogeneous deformations (strain tensor is constant)

$$\sigma_{ik} n_k = p_i \Rightarrow$$

$\sigma_{zz} = p$ and other components $= 0$

- $$u_{xx} = u_{yy} = -\frac{1}{3} \left(\frac{1}{2\mu} - \frac{1}{3K} \right) p$$

$$u_{zz} = \frac{1}{3} \left(\frac{1}{3K} + \frac{1}{\mu} \right) p = \frac{p}{E}$$

where $E = \dfrac{9K\mu}{3K+\mu}$ is the coefficient of extension or Young modulus

The ratio of the transverse compression to the longitudinal extension is called Poisson ratio

$$u_{xx} = -\sigma\, u_{zz}$$

$$\sigma = \frac{1}{2}\,\frac{3K - 2\mu}{3K + \mu}$$

Since $K, \mu > 0 \Rightarrow -1 < \sigma \leq \frac{1}{2}$

In reality, however $0 < \sigma \leq \frac{1}{2}$

$\sigma > 0 \Rightarrow \lambda > 0$

Although not required from thermodynamics $\lambda > 0$ is realized in practice (transverse compression under longitudinal stretching, exeption - crumpling membranes)

$\sigma \to \frac{1}{2}$ corresponds to the situation, when

$\mu \ll K$

In terms of E and σ free energy can be rewritten as

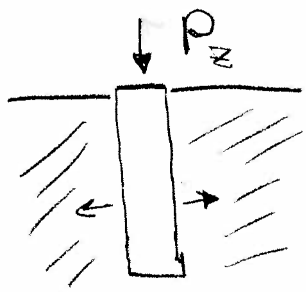
$$F = \frac{E}{2(1+\sigma)}\left(u_{ik}^2 + \frac{\sigma}{1-2\sigma}u_{ll}^2\right)$$

Then $\sigma_{ik} = \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ik}\right)$

Inverting

$$u_{ik} = \frac{1}{E}\left[(1+\sigma)\sigma_{ik} - \sigma\,\delta_{ik}\,\sigma_{ll}\right]$$

Application:



Rod with fixed sides under external pressure p_z

What is the transverse pressure it exerts on medium

Here only $u_z(z)$ is present $\Rightarrow$

$u_{zz} \neq 0$ and all the other components are $=0$

Then from the formulas above we get

$$\sigma_{xx} = \sigma_{yy} = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}u_{zz}, \quad \sigma_{zz} = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}u_{zz}$$

$$\sigma_{zz} = p \Rightarrow u_{zz} = p\,\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}$$

$$\sigma_{xx} = \sigma_{yy} = p\,\frac{\sigma}{(1-\sigma)}$$

The equation of equilibrium for isotropic bodies

Let us substitute expression for σ_{ik} in the general expression

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0 \Rightarrow$$

$$\frac{E}{2(1+\sigma)} \frac{\partial^2 u_i}{\partial x_k^2} + \frac{E}{2(1+\sigma)(1-2\sigma)} \frac{\partial^2 u_l}{\partial x_i \partial x_l} + \rho g_i = 0$$

Or in vector notations

$$\Delta \vec{u} + \frac{1}{1-2\sigma} \operatorname{grad} \operatorname{div} \vec{u} = -\rho \vec{g} \, \frac{2(1+\sigma)}{E}$$

using $\operatorname{rot} \operatorname{rot} u = \operatorname{grad} \operatorname{div} u - \Delta u$

we can exclude Δu and obtain

$$\operatorname{grad} \operatorname{div} \vec{u} - \frac{1-2\sigma}{2(1-\sigma)} \operatorname{rot} \operatorname{rot} \vec{u} = -\rho \vec{g} \, \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}$$