Two dimensional flow

In two dimensions, incompressible flow can be characterized by a single scalar function.

$$\operatorname{div} \vec{v} = 0 \Rightarrow \frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$$

Let us introduce the stream function ψ.

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}$$

Then $\operatorname{div} \vec{v} = 0$.

The streamlines are defined by

$$\frac{dx}{v_x} = \frac{dy}{v_y} \Rightarrow v_x dy - v_y dx = 0 \Leftrightarrow$$

$$d\psi = 0$$

Thus streamlines a determined by the condition $\psi = \text{const}$.

Another important use of the stream function is that the flux through any line is

$$\int_1^2 v_n \, dl = \int_1^2 (v_x dy - v_y dx) = \int_1^2 d\psi = \psi_2 - \psi_1.$$

Solid boundary has to coincide with one of the streamlines

Potential flow in 2d

$$\begin{aligned} \operatorname{div} \vec{v} &= 0 \\ \operatorname{rot} \vec{v} &= 0 \end{aligned} \quad \Rightarrow \quad v = \nabla \varphi, \qquad \nabla^2 \varphi = 0$$

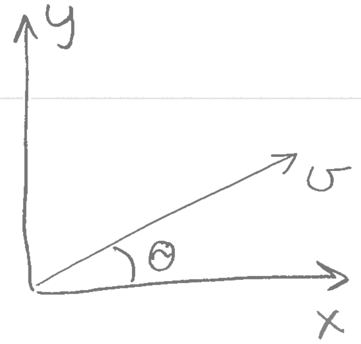
In 2d

$$v_x = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

But these are just Cauchy-Riemann conditions for the complex function $W = \varphi + i\psi$ to be an analytic function of the complex argument $z = x + iy$

Then derivative

$$\frac{dW}{dz} = \frac{1}{2}\frac{\partial W}{\partial x} - \frac{i}{2}\frac{\partial W}{\partial y} = v_x - i v_y = v e^{-i\theta}$$

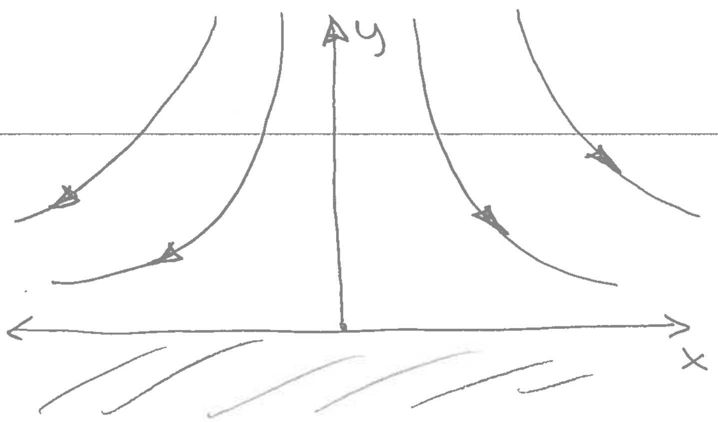


Complex flow allows one to describe some simple flows in a compact form

Uniform flow is $\underline{W = (v_x - i v_y) z}$

Another examples

1) Potential flow near a stagnation point $\vec{v} = 0$



Since at the stagnation point velocity is zero we can expand φ in Taylor series

$$\varphi = S_{ij} \frac{x_i x_j}{2}$$

$$\operatorname{div} \vec{v} = S_{ii} = 0$$

In the principal axes of the tensor one has

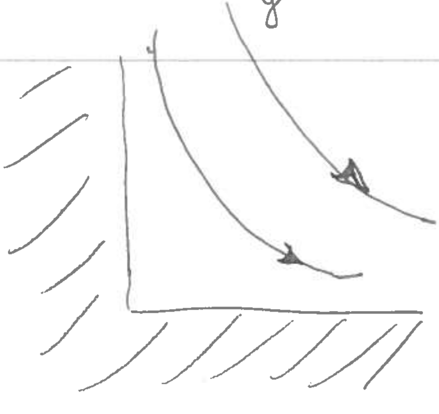
$$\varphi = k \frac{(x^2 - y^2)}{2}, \quad v_x = kx, \quad v_y = -ky \Rightarrow$$

$\psi = kxy$. All that corresponds to

$$w = \frac{k z^2}{2}$$

The streamlines are hyperbolae

Note that because of the symmetry $x \leftrightarrow y$ the same flow is valid also for another geometry

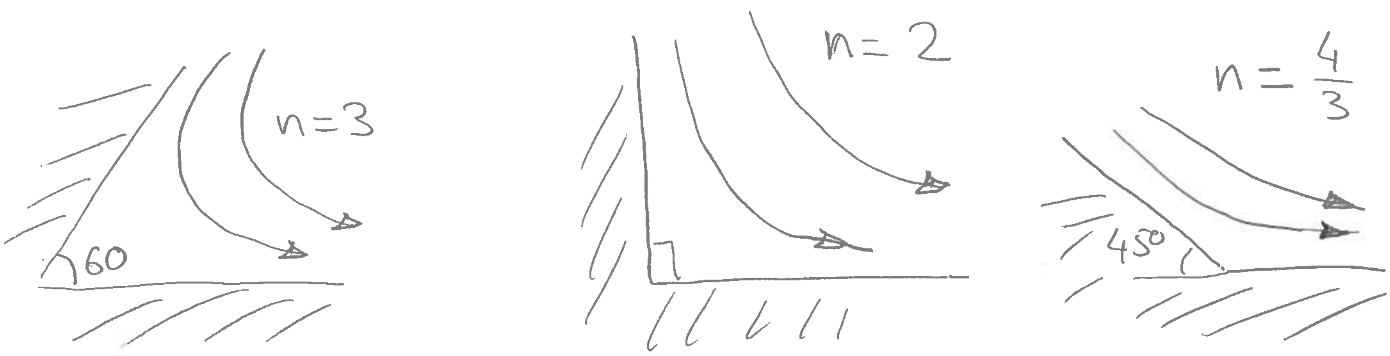


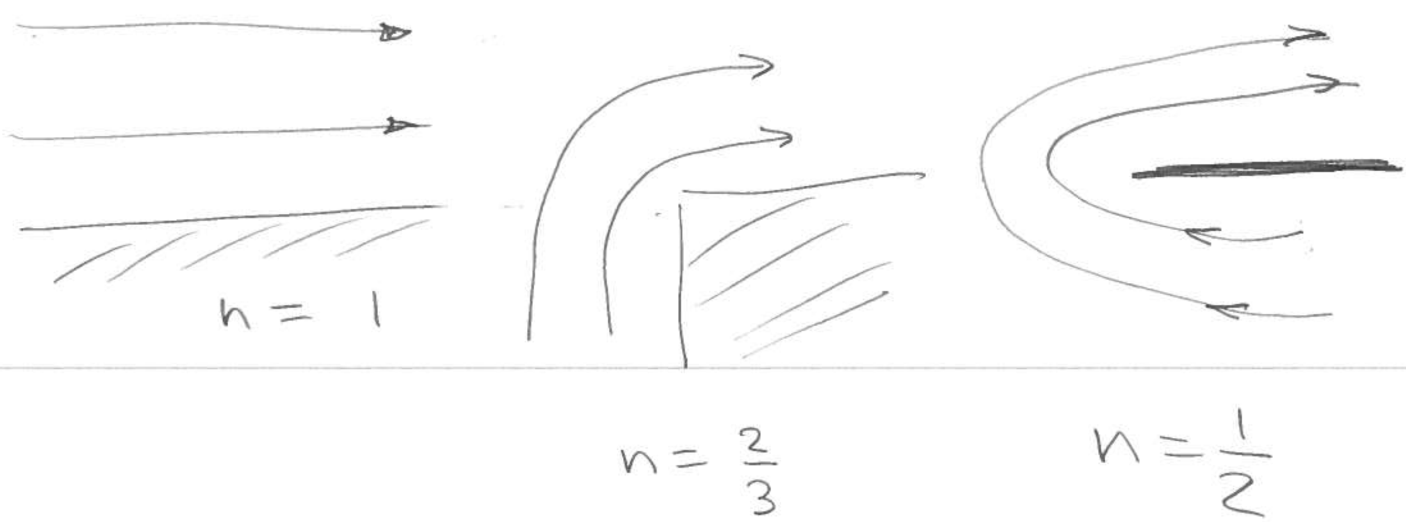
2) Consider

$$w = Az^n$$

Then $\varphi = Ar^n \cos n\varphi$, $\psi = Ar^n \sin n\varphi$

Zero flux boundary should coincide with the streamlines so two lines $\theta = 0$ and $\theta = \frac{\pi}{n}$ could be seen as boundaries





Velocity modulus $v = \left|\frac{dw}{dz}\right| = n|A|\, r^{n-1}$

at $r \to 0$ either turns to 0 $(n > 1)$ or

to infinity $(n < 1)$

One can think of those solution as obtained by <u>conformal transformation</u> $\xi = z^n$

So that in the half plane ξ the solution is the uniform flow $W = A\,\xi = A z^n$.

Using conformal mapping one can construct solutions for other geometries

Real flow usually separates at discontinuities