Quantum Field Theory II

Problem Sets

ETH Zurich, FS13

B. Mistlberger, M. Rosso, Prof. N. Beisert

Contents

1.1. Transition amplitude for a harmonic oscillator

A "classical" exercise regarding path integrals in quantum mechanics is the computation of the transition amplitude for a harmonic oscillator.

Consider the Lagrangian

$$L(q,\dot{q}) = \tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}m\omega^2 q^2, \tag{1.1}$$

where $q, \dot{q}$ are the position and velocity variables.

Compute the transition amplitude for the harmonic oscillator

$$U(q_\text{f}, q_\text{i}, t) := \langle q_\text{f}, t_\text{f} | q_\text{i}, t_\text{i}\rangle = \int Dq \, \exp\left[\frac{i}{\hbar}\int_{t_\text{i}}^{t_\text{f}} dt\, L(\hat{q},\hat{\dot{q}})\right], \tag{1.2}$$

where $t := t_\text{f} - t_\text{i}$. Show that it equals

$$U(q_\text{f}, q_\text{i}, t) = \left[\frac{m\omega}{2\pi\hbar i \sin(\omega t)}\right]^{1/2} \exp\left[\frac{im\omega}{2\hbar}\,\frac{\cos(\omega t)(q_\text{i}^2 + q_\text{f}^2) - 2q_\text{i}q_\text{f}}{\sin(\omega t)}\right]. \tag{1.3}$$

Walkthrough:

a) Split up the path $q(t)$ in n intermediate steps q_k, $k = 1, \ldots, n-1$,

$$U(q_\text{f}, q_\text{i}, t) = \lim_{n\to\infty}\int\left[\prod_{j=1}^{n-1} dq_j\right]\left[\frac{nm}{2\pi\hbar i t}\right]^{n/2}\exp\left[\frac{i}{\hbar}\sum_{k=1}^{n}\frac{t}{n}L_k\right], \tag{1.4}$$

where $q_0 := q_\text{i}$, $q_n := q_\text{f}$ and $L_k = \langle q_{k+1}|L(\hat{q},\hat{\dot{q}})|q_k\rangle$.

Write the Lagrangian expectation value L_k in terms of the q_k. Choose a suitable definition of "ordering" of the operators with

$$\langle q_k|\hat{\dot{q}}|q_{k-1}\rangle = \frac{q_k - q_{k-1}}{t/n}; \tag{1.5}$$

make a wise choice for $V(\hat{q})$.

b) Reexpress the exponential in (1.4) as a Gaussian function of the form

$$\exp\left[\frac{i}{\hbar}\sum_{k=1}^{n}\frac{t}{n}L_k\right] = \exp\left[\frac{inm}{2\hbar t}\left(\vec{q}^{\,\intercal} M_{n-1}\vec{q} + \vec{B}^{\intercal}\vec{q}\right)\right], \tag{1.6}$$

where $\vec{q} = (q_1, q_2, \ldots, q_{n-1})$ is an $(n-1)$-dimensional vector, M is an $(n-1)\times(n-1)$ matrix, $\vec{B}$ is an $(n-1)$-dimensional vector (depending on $q_0 = q_\text{i}$, $q_n = q_\text{f}$). With this form, you should be able to compute the integral (rather) easily.

⟶

c) Show that the determinant of $D_{j-1} := \det M_{j-1}$ satisfies the equality

$$\frac{D_{j+1} + D_{j-1} - 2D_j}{(t/n)^2} = -a^2 D_j \tag{1.7}$$

when considered as a function of the steps j. Determine the constant a, then solve the equation.

Hint: this equation is the discretised version of a well-known equation. In order to solve it, you can solve the continuum equations for $D(t)$, re-express the solution in discretised time and take the leading limit as $n \to \infty$, with b.c. $D_0 = 0$, $D_1 = 1 + \mathcal{O}(1/n^2)$. Why do we choose these b.c.?

Check: you should get that

$$D_{n-1} \sim \frac{n}{\omega t} \sin(\omega t) \quad \text{for } n \to \infty \tag{1.8}$$

d) Now, if you did everything correctly, you should be left with the computation of the coefficients of $(q_\mathrm{i}^2 + q_\mathrm{f}^2)$ and of $q_\mathrm{i} q_\mathrm{f}$ in the exponent. You can compute them by computing the appropriate minors of the matrix M_{n-1} and get the correct result. Enjoy!

2.1. Propagator of a free Klein–Gordon field

The aim of this exercise is to show that the Feynman propagator of a free Klein–Gordon field yields the usual expression, well known from QFT I, when derived using the path integral formalism

$$\begin{aligned}\langle 0|\mathrm{T}(\phi(x_1)\phi(x_2))|0\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\int D\phi\,\phi(x_1)\phi(x_2)\exp\bigl(i\int_{-T}^{+T} d^4x\,\mathcal{L}\bigr)}{\int D\phi \exp\bigl(i\int_{-T}^{+T} d^4x\,\mathcal{L}\bigr)} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{-ie^{-ik(x_1-x_2)}}{k^2+m^2-i\epsilon}. \end{aligned} \tag{2.1}$$

a) Show that the following integral is given by

$$\int d^n x\ \exp\bigl(-x^{\mathsf{T}} M x\bigr) = \frac{\text{const.}}{\sqrt{\det(M)}}, \tag{2.2}$$

where the x_i are the components of an n-dimensional Euclidean vector x and M is a symmetric matrix. Carry out the integrals by diagonalising the matrix M and redefining the vector x. Compute similarly

$$\int d^n x\, x_j^2 \exp\bigl(-x^{\mathsf{T}} M x\bigr), \tag{2.3}$$

where x_j is one of the components of the vector x. Express the result as a factor times the integral in (2.2).

To compute (2.1) we discretise spacetime

$$\int D\phi \to \int_{k_n^0>0} d\,\mathrm{Re}(\phi(k_n))\,d\,\mathrm{Im}(\phi(k_n)), \qquad \int \frac{d^4k}{(2\pi)^4} \to \frac{1}{L^4}\sum_n . \tag{2.4}$$

You can think of this discretisation as regarding the 4-dimensional spacetime on a lattice of size L with lattice spacing, i.e. the distance between two spacetime points, L/N. In the continuum limit, the lattice spacing goes to zero, and the lattice size goes to infinity

$$L/N \to 0, \qquad L\to\infty. \tag{2.5}$$

The mode-vector is $k_n^\mu = 2\pi n^\mu/L$, where the components n^μ are integers between $-N/2$ and $N/2$. We will take the continuum limit at the end. Also we perform a discrete mode expansion of the free scalar field

$$\phi(x) = \frac{1}{L^4}\sum_n e^{-ik_n x}\phi(k_n). \tag{2.6}$$

The individual Fourier coefficients $\phi(k_n)$ are complex but the field $\phi(x)$ is real. Thus, we have the constraint $\phi(k_n)^* = \phi(-k_n)$. However, we can treat the real and imaginary part of $\phi(k)$ as independent variables if we restrict ourselves to modes with $k_n^0 > 0$.

⟶

b) Insert the mode expansion of $\phi(x)$ and find the discretised equivalent to the action of the Klein–Gordon field.

$$S = \int d^4x \left(-\tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi(x)^2\right). \tag{2.7}$$

The result is quadratic in the real and imaginary part of $\phi(k_n)$.

c) Compute the discrete equivalent of

$$\int D\phi\, e^{iS} = \prod_n \sqrt{\frac{-i\pi L^4}{k_n^2 + m^2}}\,. \tag{2.8}$$

Separate the integrations over the imaginary and real part of ϕ and make use of the results obtained in problem a). When recombining both parts you should find the correct result.

d) Now compute the discretised version of (2.1) by inserting the mode expansion for $\phi(x_1)\phi(x_2)$. Make use of the symmetry of the integrand (even or odd in ϕ) to maintain only non-vanishing terms. Can you relate your expressions to what you found in problem a)? Finally, take the continuum limit and recover the Feynman propagator.

3.1. Generating functionals

Consider a Lagrangian for massless scalars with a quartic interaction

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad \mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \quad \mathcal{L}_{\text{int}} = -\tfrac{1}{24}\lambda\phi^4. \tag{3.1}$$

The interacting generating functional is then

$$Z[j] = \exp\left[i\int d^4x\,\mathcal{L}_{\text{int}}\left(\frac{-i\delta}{\delta j(x)}\right)\right] Z_0[j], \tag{3.2}$$

expressed in terms of the free generating functional

$$Z_0[j] = \int D\phi\,\exp\left[i\int d^4x\,\big(\mathcal{L}_0 + j(x)\phi(x)\big)\right]. \tag{3.3}$$

a) Show that

$$Z_0[j] = Z_0[0]\exp\left[\frac{i}{2}\int d^4y_1\,d^4y_2\,j(y_1)G_{\text{F}}(y_1,y_2)j(y_2)\right], \tag{3.4}$$

where $G_{\text{F}}(x-y)$ is exactly the *Feynman* propagator (and not just any Green function).

b) Compute the vacuum contributions to $Z[j]$ to order λ^2,

$$\begin{aligned}\frac{Z[0]}{Z_0[0]} &= \frac{1}{Z_0[0]}\exp\left[-i\int d^4x\,\mathcal{L}_{\text{int}}\left(\frac{-i\delta}{\delta j(x)}\right)\right] Z_0[j]\bigg|_{j=0}\\ &= 1 + \lambda\int dx\,C_1 G_{\text{F}}(x,x)^2\\ &\quad + \lambda^2\int dx\,dy\,\Big[C_{2,1}\,G_{\text{F}}(x,x)^2 G_{\text{F}}(y,y)^2\\ &\qquad + C_{2,2}G_{\text{F}}(x,x)G_{\text{F}}(y,y)G_{\text{F}}(x,y)^2 + C_{2,3}\,G_{\text{F}}(x,y)^4\Big] + \mathcal{O}(\lambda^3),\end{aligned} \tag{3.5}$$

being particularly careful in the computation of the combinatorial factors C_1, $C_{2,1}$, $C_{2,2}$, $C_{2,3}$. These factors can be computed either from the functional derivative expression or as symmetry factors of the related diagrams.

Describe graphically each term and identify connected and disconnected terms. Then show that the functional $W[0]$ with

$$W[j] := -i\log\left[\frac{Z[j]}{Z_0[0]}\right] \tag{3.6}$$

generates only the connected contributions to the vacuum amplitude.

⟶

3.2. Berezin integral

Throughout this exercise, the symbol θ_j will indicate an odd (i.e. anti-commuting) *real* variable. We will study some general properties of Berezin integration over functions $f(\theta)$.

a) We know from the lecture that the fermionic delta function for a single odd variable can be written as

$$\delta(\theta - \eta) = \theta - \eta. \tag{3.7}$$

In order to generalise this notion, we first have to define the notion of multidimensional Berezin integral. Given a set of odd variables $\theta_1, \theta_2, \ldots, \theta_n$, we define the n-dimensional Berezin measure as

$$d^n\theta := d\theta_n \, d\theta_{n-1} \, \ldots \, d\theta_1, \tag{3.8}$$

so that the following identity holds:

$$\int d^n\theta \, \theta_1 \, \theta_2 \, \ldots \, \theta_n = 1. \tag{3.9}$$

Define the multidimensional delta function as

$$\delta^{0|n}(\theta - \eta) = \prod_{i=1}^{n} (\theta_i - \eta_i) = (\theta_1 - \eta_1) \ldots (\theta_n - \eta_n). \tag{3.10}$$

Show that, with these definitions,

$$\int d^n\theta \, \delta^{0|n}(\theta - \eta) \, f(\theta) = f(\eta). \tag{3.11}$$

b) Consider a function $f(\theta)$, where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. Show that, under a non-singular linear transformation (M is real and Grassmann even)

$$\eta = M\theta, \tag{3.12}$$

the following identity is satisfied:

$$\int d^n\eta \, f(\eta) = \frac{1}{\det(M)} \int d^n\theta \, f(\theta), \tag{3.13}$$

meaning that, under a change of variable, a Berezin integral (over purely odd coordinates) transforms with the *inverse* Jacobian.

c) Consider the integral

$$\int d^n\theta \, \exp\left(\tfrac{1}{2}\theta_i A_{ij} \theta_j\right), \tag{3.14}$$

where A is an $n \times n$ real, skew-symmetric matrix. Compute the integral for n even. Show that it vanishes for n odd.

Hint: recall that any real, anti-symmetric matrix can be brought to a block diagonal form via an orthogonal transformation.

4.1. Schwinger–Dyson equation

Consider a free massive scalar field with Lagrangian

$$\mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2. \tag{4.1}$$

The path integral is invariant under a linear shift of the field $\phi(x) \to \phi(x) + \epsilon(x)$. Use this property to show that the correlation functions satisfy the classical equations of motion up to contact terms, i.e.

$$(-\partial^2 + m^2)\langle\phi(x)\phi(x_1)\ldots\phi(x_n)\rangle = -i\sum_{k=1}^{n}\langle\phi(x_1)\ldots\delta^4(x-x_k)\ldots\phi(x_n)\rangle. \tag{4.2}$$

4.2. Generating functionals

Consider now the Lagrangian of a massive scalar field ϕ with quartic interaction

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad \mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = -\tfrac{1}{24}\lambda\phi^4. \tag{4.3}$$

The generating functional with sources $j(x)$ is then

$$Z[j] = Z_0[0]\exp\left[i\int d^4x\,\mathcal{L}_{\text{int}}\left(\frac{-i\delta}{\delta j(x)}\right)\right]\exp\left[\frac{i}{2}\int d^4y_1 d^4y_2\, j(y_1)G_{\text{F}}(y_1-y_2)j(y_2)\right], \tag{4.4}$$

where $G_{\text{F}}(x)$ is the Feynman propagator. In the last exercise sheet you convinced yourself that $Z[j]$ produces vacuum contributions. Furthermore, you saw that the functional

$$W[j] = -i\log Z[j] \tag{4.5}$$

could be used to generate only the connected graphs.

a) Compute $Z[j]$ and $W[j]$ at $\mathcal{O}(\lambda)$ and $\mathcal{O}(j^4)$, i.e. drop any contributions with more than one vertex or more than four sources. *Hint:* The entire exercise can be solved graphically.

b) Convince yourself graphically that the contributions to the four-point correlation function at $\mathcal{O}(\lambda)$ arise correctly from

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = Z[j]^{-1}\frac{\delta}{\delta j(x_1)}\frac{\delta}{\delta j(x_2)}\frac{\delta}{\delta j(x_3)}\frac{\delta}{\delta j(x_4)}Z[j]\bigg|_{j=0} \tag{4.6}$$

and that the connected contributions arise from

$$\frac{\delta}{\delta j(x_1)}\frac{\delta}{\delta j(x_2)}\frac{\delta}{\delta j(x_3)}\frac{\delta}{\delta j(x_4)}W[j]\bigg|_{j=0}. \tag{4.7}$$

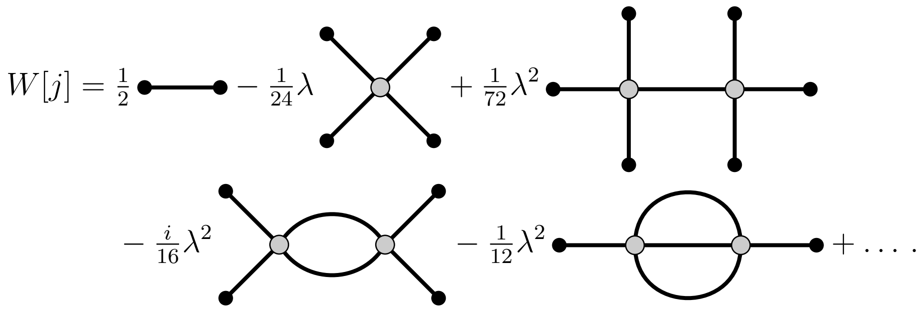
c) Determine the connected contributions at $\mathcal{O}(\lambda^2)$ with four external legs.

$\longrightarrow$

4.3. Effective action

Graphically compute the effective action $G[\phi]$ for the above model defined by (4.3) at $\mathcal{O}(\lambda^2)$ ignoring vacuum bubbles and tadpoles.

You may start with $W[j]$ given by the graphs



(4.8)

Throughout the calculation you should ignore any vacuum bubble graph, tadpole graph and graph with more than two interaction vertices.

a) Compute the field functional $\phi[j] = \delta W[j]/\delta j$.

b) Determine the inverse functional $j[\phi]$ by demanding that $j[\phi[j]] = j$ or $\phi[j[\phi]] = \phi$.
Hint: Use the result you obtained in part a). Substitute graphically the result you obtained at leading order in λ into the all higher order terms, and continue like this order by order in λ. Immediately drop terms of $\mathcal{O}(\lambda^3)$.

c) Compute $W[j[\phi]]$ and $S_{\text{src}}[\phi, j[\phi]] = \int d^4x\, \phi(x) j[\phi](x)$ by graphically substituting the result you obtained for $j[\phi]$ in b).

d) Take the difference $G[\phi] = W[j[\phi]] - S_{\text{src}}[\phi, j[\phi]]$. Convince yourself that all remaining terms in $G[\phi]$ are 1PI.

5.1. Lie Algebras

An algebra is a vector space $\mathbb{A}$ over a field $\mathbb{K}$ together with a bilinear operation $\mathbb{A}\times\mathbb{A}\to\mathbb{A}$. A Lie algebra $\mathfrak{g}$ is an algebra whose operator is a so-called Lie bracket denoted by $[\cdot,\cdot]_{\text{Lie}}$. A Lie bracket must satisfy the properties ($A,B,C\in\mathfrak{g}$ and $a,b\in\mathbb{K}$)

- bilinearity

$$\begin{aligned}[aA+bB,C]_{\text{Lie}} &= a[A,C]_{\text{Lie}}+b[B,C]_{\text{Lie}},\\ [C,aA+bB]_{\text{Lie}} &= a[C,A]_{\text{Lie}}+b[C,B]_{\text{Lie}},\end{aligned} \tag{5.1}$$

- anti-symmetry

$$[A,B]_{\text{Lie}} = -[B,A]_{\text{Lie}}, \tag{5.2}$$

- Jacobi identity

$$[A,[B,C]_{\text{Lie}}]_{\text{Lie}}+[C,[A,B]_{\text{Lie}}]_{\text{Lie}}+[B,[C,A]_{\text{Lie}}]_{\text{Lie}}=0. \tag{5.3}$$

The matrix algebras $\mathfrak{sl}(N)$, $\mathfrak{so}(N)$ and $\mathfrak{sp}(N)$ are equipped with the matrix commutator as bilinear operation, $[\cdot,\cdot]_{\text{Lie}}=[\cdot,\cdot]$. The elements of the algebras are

- $\mathfrak{sl}(N)$: traceless $N\times N$ matrices,
- $\mathfrak{so}(N)$: $N\times N$ matrices that are anti-symmetric w.r.t. a non-degenerate symmetric metric M, $A^{\mathsf{T}}=-MAM^{-1}$,
- $\mathfrak{sp}(N)$: $N\times N$ matrices that are anti-symmetric w.r.t. a non-degenerate anti-symmetric matrix E, $A^{\mathsf{T}}=-EAE^{-1}$.

Show that the constraints for $\mathfrak{so}(N)$ and $\mathfrak{sp}(N)$ are self-consistent, i.e. compare A^{TT} to A. Show that these matrix algebras are indeed algebras and Lie algebras.

5.2. Quadratic Casimir Invariant

Consider the basis T_a of generators of a simple Lie algebra $\mathfrak{g}$. One can define a unique invariant symmetric bilinear form called Killing form satisfying

$$K(T_a,T_b)=K(T_b,T_a),\qquad K([T_a,T_b]_{\text{Lie}},T_c)+K(T_a,[T_b,T_c]_{\text{Lie}})=0. \tag{5.4}$$

The Killing form can be constructed as a trace of two generators in some representation

$$K(T_a,T_b)=k_{ab},\qquad \operatorname{Tr}(R(T_a)R(T_b))=B^R k_{ab}, \tag{5.5}$$

where B^R is a representation-dependent proportionality factor. For a semi-simple Lie algebra the Killing form is invertible and we can find a central element $C_2:=k^{ab}T_aT_b$ of the enveloping algebra $\mathrm{U}(\mathfrak{g})$ that commutes with all elements of $\mathrm{U}(\mathfrak{g})$.

Show that for an irreducible representation R, the dimension D^R of R, the dimension D^{ad} of the adjoint representation and the eigenvalue C_2^R characterising $R(C_2)=C_2^R\,\mathrm{id}^R$ are related by

$$D^{\text{ad}}B^R=D^RC_2^R. \tag{5.6}$$

$\longrightarrow$

5.3. Tensor Product Representation

Representations of a Lie algebra can be combined to give bigger representations via direct sums

$$R_{1\oplus 2}(a) = \begin{pmatrix} R_1(a) & 0 \\ 0 & R_2(a) \end{pmatrix} \tag{5.7}$$

and tensor products

$$R_{1\otimes 2}(a) = R_1(a) \otimes \mathrm{id}_2 + \mathrm{id}_1 \otimes R_2(a). \tag{5.8}$$

a) Check that the tensor product of two representations is again a representation. Here R_1, R_2 are proper representations of $\mathfrak{g}$.

b) Define a permutation P on the tensor product of two identical vector spaces $\mathbb{V} \otimes \mathbb{V}$ by

$$P(v_1 \otimes v_2) = v_2 \otimes v_1. \tag{5.9}$$

Then the projectors onto the symmetric and anti-symmetric sub-spaces $\mathbb{V}^\pm \subset \mathbb{V} \otimes \mathbb{V}$ are given by $P^\pm = \frac{1}{2}(\mathrm{id} \pm P)$. Show that this projector allows to rearrange the tensor product of two identical representations as the direct sum of the symmetric and anti-symmetric representation $R^\pm$

$$R \otimes R = R^+ \oplus R^-, \qquad R^\pm = P^\pm (R \otimes R) P^\pm. \tag{5.10}$$

c) More concretely: Define a basis v_α for the space $\mathbb{V}$, and write the representation R as a matrix in this basis, i.e.

$$R(a) v_\alpha = R(a)_\alpha{}^\beta v_\beta. \tag{5.11}$$

Can you now write the components of $R \otimes R$ and $R^\pm$ in the basis $v_\alpha \otimes v_\beta$ of $\mathbb{V} \otimes \mathbb{V}$?

6.1. Completeness relation and Casimirs for $\mathfrak{su}(N)$

The special unitary algebra $\mathfrak{su}(N)$ is defined as the commutator algebra on the space

$$\mathfrak{su}(N) := \{N \times N \text{ anti-hermitian traceless matrices}\}. \tag{6.1}$$

Let $T_a = T_a^{\text{def}}$, $a = 1, \ldots N^2 - 1$, be an imaginary basis for this vector space $\mathfrak{su}(N) = \text{span}\{iT_a\}$, or, in other words, a basis for *hermitian* traceless matrices.
The (non-degenerate) Killing metric k_{ab} can be obtained from the T_a^{def} as

$$\operatorname{tr}(T_a^{\text{def}} T_b^{\text{def}}) = B^{\text{def}} k_{ab}. \tag{6.2}$$

The structure constants $f_{ab}{}^c$ are defined as usual, $[T_a, T_b] =: i f_{ab}{}^c T_c$. For $\mathfrak{su}(N)$ there is a similar set of *symmetric* constants $d_{ab}{}^c$ obtained from the defining representation

$$\{T_a^{\text{def}}, T_b^{\text{def}}\} =: d_{ab}{}^c T_c^{\text{def}} + 2\,\frac{B^{\text{def}}}{N}\,k_{ab}. \tag{6.3}$$

The quadratic and cubic Casimir elements (of the enveloping algebra) are defined as

$$C_2 := k^{ab} T_a T_b, \qquad C_3 := d^{abc} T_a T_b T_c. \tag{6.4}$$

a) Let X be a generic $N \times N$ complex matrix. Prove the completeness relation

$$k^{ab} \operatorname{tr}(T_a^{\text{def}} X) T_b^{\text{def}} = B^{\text{def}} \left(X - \frac{1}{N} \operatorname{tr}(X)\,\text{id} \right). \tag{6.5}$$

Hint: consider the space of $N \times N$ complex matrices as a N^2-dimensional vector space over $\mathbb{C}$ and find a suitable basis by extending the basis of $\mathfrak{su}(N)$.

b) Let X be a generic $N \times N$ complex matrix. Knowing the previous identity, eq. (6.5), prove the completeness relation

$$k^{ab} T_a^{\text{def}} X T_b^{\text{def}} = B^{\text{def}} \left(\operatorname{tr}(X)\,\text{id} - \frac{1}{N} X \right). \tag{6.6}$$

c) Show that the symmetric structure constants $d_{ab}{}^c$ are traceless in the first two indices,

$$k^{ab} d_{ab}{}^c = 0. \tag{6.7}$$

d) Show that the Casimir invariants $k^{ab} T_a^{\text{def}} T_b^{\text{def}} = C_2^{\text{def}}\,\text{id}^{\text{def}}$ and $d^{abc} T_a^{\text{def}} T_b^{\text{def}} T_c^{\text{def}} = C_3^{\text{def}}\,\text{id}^{\text{def}}$ for the defining representation are given by

$$C_2^{\text{def}} = \frac{N^2 - 1}{N} B^{\text{def}}, \qquad C_3^{\text{def}} = \frac{(N^2 - 4)(N^2 - 1)}{N^2} (B^{\text{def}})^2. \tag{6.8}$$

$\longrightarrow$

6.2. Simple Lie groups and simple Lie algebras

You know that a simple Lie group is a connected non-abelian Lie group with no proper connected normal subgroups. We want to understand what this condition means in terms of the Lie algebra of the group. In order to do this, we will have to introduce some additional algebraic notion. In this exercise, we will always consider simply connected Lie groups, so that the exponential map *is* a covering map.

a) A *subalgebra* is a subset of an algebra which is closed under multiplication. In the case of a Lie algebra, this means that, given an algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ obeys

$$\mathfrak{h} \subseteq \mathfrak{g} \,:\, [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}. \tag{6.9}$$

Show that the exponential map maps a subalgebra $\mathfrak{h}$ into a subgroup H of G.
Hint: recall the Baker–Campbell–Hausdorff formula

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big([X,[X,Y]] - [Y,[X,Y]]\big) + \dots\right), \tag{6.10}$$

where the dots denote an infinite sum of nested Lie brackets of X, Y.

b) An *ideal* is a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with the property that

$$[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}. \tag{6.11}$$

A proper ideal is a nonempty ideal not coincident with the whole algebra $\mathfrak{g}$.
Show that if a group is simple, its Lie algebra contains no proper ideals.
Hint: it is enough to work with infinitesimal elements of the group, i.e.

$$a := \exp(\epsilon A) \simeq \mathrm{id} + \epsilon A + \mathcal{O}(\epsilon^2). \tag{6.12}$$

6.3. Analysis of $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$

Consider the defining representations of the real Lie algebras $\mathfrak{so}(3)$, $\mathfrak{so}(2,1)$:

$$\begin{aligned} \mathfrak{so}(3) &:= \{M \in \mathbb{R}^{3\times 3} \,:\, M I_3 + I_3 M^\intercal = 0\}, \\ \mathfrak{so}(2,1) &:= \{M \in \mathbb{R}^{3\times 3} \,:\, M I_{2,1} + I_{2,1} M^\intercal = 0\}, \end{aligned} \tag{6.13}$$

where

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad I_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.14}$$

a) Choose a (sensible) basis for the defining representation of both algebras and show that they are simple Lie algebras.

b) Compute the Killing form for both algebras. How do you explain the difference in the signature?

c) Compute the Casimir invariants for $\mathfrak{so}(3)$ in the defining and in the adjoint representation.

7.1. SU(N) gauge theory

Let us introduce a group-valued smooth function $U(x, y)$, called comparator, that allows to relate the gauge phase of fields at different points in spacetime. The comparator satisfies $U(x, x) = 1$ and transforms under a gauge transformation $V(x)$ as

$$U(x, y) \to V(x)U(x, y)V^{-1}(y). \tag{7.1}$$

We can use the comparator to define a covariant derivate by

$$n^\mu D_\mu \psi(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \big[\psi(x + n\epsilon) - U(x + n\epsilon, x)\psi(x)\big], \tag{7.2}$$

where n^μ is an arbitrary vector and $\psi(x)$ is a field. Expanding the comparator around unity yields

$$U(x + \epsilon n, x) = 1 + i\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2). \tag{7.3}$$

a) Consider $V(x)$ to be an abelian gauge transformation. How does the gauge field $A_\mu(x)$ have to transform such that D_μ is truly a covariant derivative satisfying

$$V(x)D_\mu\psi(x) = D'_\mu V(x)\psi(x)? \tag{7.4}$$

b) You can construct a gauge-invariant quantity by using the comparator to connect a space-time point to itself in a non-trivial way

$$W(x) = U(x, x + \epsilon n)U(x + \epsilon n, x + \epsilon n + \epsilon m)U(x + \epsilon n + \epsilon m, x + \epsilon m)U(x + \epsilon m, x). \tag{7.5}$$

Expand $W(x)$ and compare the $\mathcal{O}(\epsilon^2)$ term to the abelian field-strength $[D_\mu, D_\nu] = iF_{\mu\nu}$.

Consider now $V(x)$ to be a non-abelian gauge transformation.

c) How does the non-abelian gauge field transform under a gauge transformation such that the definition of the covariant derivative (7.2) is unchanged?

d) How does the non-abelian field-strength tensor $[D_\mu, D_\nu] = iF_{\mu\nu}$ transform under a gauge transformation? Compare to the abelian case.

e) Show that the quantity $\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ is gauge invariant.

⟶

7.2. Chromodynamics

The chromodynamics Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2g_{\mathrm{YM}}^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}), \tag{7.6}$$

where $F_{\mu\nu}$ is the field-strength tensor of a SU(3) gauge field A_μ in the defining representation and ψ is 3-vector of Dirac fields.

a) Write down the equations of motion for the fields.

b) Show that the fermionic current $(J^\mu)^\alpha{}_\beta = -\bar{\psi}_\beta\gamma^\mu\psi^\alpha$ is covariantly conserved

$$[D_\mu, J^\mu] = 0. \tag{7.7}$$

c) Expand the terms in the Lagrangian in terms of the gauge field $A_\mu = gT_a^{\mathrm{def}}A_\mu^a$ and interpret pictorially the individual terms.

d) Can you write down a Lagrangian for a scalar field that is invariant under SU(3)?

8.1. Path integral gauge fixing

We will consider some gauge fixing different from the standard Landau gauge seen during the lecture

a) Consider the action for pure electrodynamics

$$S_{\text{ED}} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{8.1}$$

Perform the gauge fixing via the Faddeev–Popov method, using the nonlinear gauge condition

$$G[A,\Omega] = \partial_\mu A^\mu + \zeta A_\mu A^\mu - \Omega. \tag{8.2}$$

Compute the propagator for the photon field and for the ghost field (it is enough to invert the kinetic operators appearing in the action).

Is the ghost field decoupled in this gauge? Show that in the limit $\zeta \to 0$ the Lorenz gauge is restored.

b) Consider the action for pure Yang–Mills theory

$$S_{\text{YM}} = \int d^4x \left[-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}\right]. \tag{8.3}$$

Perform the gauge fixing via the Faddeev–Popov method, using the *axial* gauge condition

$$G[A,\Omega]^a = n^\mu A^a_\mu - \Omega^a, \tag{8.4}$$

where n^μ is some fixed four vector. The possible choices are

- $n^2 = 0$, light-light gauge, e.g. $n^\mu = (1,0,0,1)$;
- $n^2 > 0$, spatial gauge, e.g. $n^\mu = (0,0,0,1)$;
- $n^2 < 0$, temporal gauge, e.g. $n^\mu = (1,0,0,0)$.

Compute the propagator for the photon field and for the ghost field (it is enough to invert the kinetic operators appearing in the action).

Does the ghost field decouple from gauge fields in this gauge?

⟶

8.2. Passarino–Veltman reduction

We will study a method to reduce one loop tensor integrals to linear combinations of one loop scalar integral. The integrals are defined in D dimension. Specifically, we define the integrals as

$$A_0(m^2) := \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2}, \tag{8.5}$$

$$B_{0,\mu,\mu\nu}(p; m_1^2, m_2^2) := \int \frac{d^D k}{(2\pi)^D} \frac{1, k_\mu, k_\mu k_\nu}{[k^2 + m_1^2][(k-p)^2 + m_2^2]}. \tag{8.6}$$

For simplicity, we have omitted the $i\epsilon$ term from the propagators. Furthermore, we shall omit the arguments of the scalar functions B which are always $(p; m_1^2, m_2^2)$.

The idea is to reduce the integrals $B_\mu, B_{\mu\nu}$ to a linear combination of A_0, B_0 integrals with suitable coefficients. The first step is to decompose the tensor integrals according to the Lorentz structure; the only allowed possibility for the two tensor integrals is

$$B_\mu = p_\mu B_1, \tag{8.7}$$

$$B_{\mu\nu} = p_\mu p_\nu B_{21} + \eta_{\mu\nu} B_{22}, \tag{8.8}$$

where B_1, B_{21}, B_{22} are to-be-determined linear combinations of A_0's and B_0's.

a) Express B_1 in eq. (8.7) as a linear combination of $A_0(m_1^2)$, $A_0(m_2^2)$ and B_0.
Hint: Write B_μ as the integral of eq. (8.6), multiply both sides of eq. (8.7) times p^μ and then rewrite the numerator of the integrand as a linear combination of m_1^2, m_2^2, p^2 as well as the factors in the denominator of the integrand.[1]
Check that

$$2p^2 B_1 = A_0(m_1^2) - A_0(m_2^2) + (m_1^2 - m_2^2 - p^2) B_0. \tag{8.9}$$

b) For the rank-2 tensor integrals, we can obtain a 2×2 linear system of equations by multiplying both sides of eq. (8.8) times $\eta^{\mu\nu}$ and p^μ, respectively (recall that we are working with dimensional regularisation).

- Repeat the steps for the manipulation of the numerator of the integrands. Pay attention when handling the numerator of the equation obtained as p^ν times eq. (8.8), (you should be able to express the RHS in terms of B_μ and of $p_\mu A_0$).
  N.B.: recall the integral

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{k^2 + m^2} = 0. \tag{8.10}$$

- Show that the linear system expressing B_{21}, B_{22} in terms of the scalar integrals A_0, B_0 is

$$p^2 B_{21} + D B_{22} = X_1 A_0(m_2^2) + X_2 B_0, \tag{8.11}$$

$$p^2 B_{21} + B_{22} = Y_1 A_0(m_2^2) + Y_2 B_1, \tag{8.12}$$

  where X_1, X_2, Y_1, Y_2 depend on m_1, m_2, p^2. Determine X_1, X_2, Y_1, Y_2.

[1] The reduction of the numerator of the integrand to a sum of m_1^2, m_2^2, p^2 and the denominators is generally possible only with one loop integrals; at higher loops, the reduction to scalar integrals is much more complicated.

9.1. Feynman's parametrisation of loop integrals

Prove the following identity, which is used to combine the denominators of integrands in loop integrals:

$$\frac{1}{D_1^{a_1}\dots D_n^{a_n}} = \frac{\Gamma(\sum_{j=1}^n a_j)}{\Gamma(a_1)\dots\Gamma(a_n)} \int_0^1 dx_1 \dots \int_0^1 dx_n \, \frac{\delta(1-\sum_{j=1}^n x_j)\, x_1^{a_1}\dots x_m^{a_n}}{[x_1 D_1 + \dots + x_n D_n]^{a_1+\dots+a_n}} \,. \tag{9.1}$$

Hint: carry out the proof by induction, starting from $n = 2$; to prove the $n = 2$ case, you will need the following definition for the Gaussian hypergeometric function ${}_2\mathrm{F}_1$ (for $\mathrm{Re}(c) > \mathrm{Re}(b), \mathrm{Re}(a) > 0$, $|z| < 1$):

$$\begin{aligned} {}_2\mathrm{F}_1(a,b;c;z) &:= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dx\, x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!} \,. \end{aligned} \tag{9.2}$$

Hint: show that

$${}_2\mathrm{F}_1(a+b,a;a+b;z) = (1-z)^{-a}. \tag{9.3}$$

9.2. Renormalisation of ϕ^3 scalar theory

Consider a scalar theory in $D = 6$ with cubic interaction, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{6}\phi^3. \tag{9.4}$$

a) Derive the Feynman rules for the theory. Show, by power counting, that the theory is renormalisable in $D = 6$.

b) Compute the one-loop self energy of the field in dimensional regularisation.

Recall that the theory is formulated in $D = 6$, so we regularise UV divergencies by computing the diagram in $D = 6 - 2\epsilon$ dimensions, $\epsilon > 0$.

c) Identify the divergent contributions in the one-loop self energy; compute the needed mass and field renormalisation counterterms that renormalise the divergencies of the self energy.

d) Compute the one loop vertex contribution. Identify the divergent part, and compute the required counterterm.

10.1. Asymptotic symmetries

Consider a theory (in $D = 4$) with two interacting real scalar fields ϕ_1, ϕ_2, with Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 - \tfrac{1}{24}\lambda(\phi_1^4 + \phi_2^4) - \tfrac{1}{4}\rho\,\phi_1^2\phi_2^2, \tag{10.1}$$

with $\lambda > 0$ and $3\rho \geq -\lambda$.

We want to compute the one-loop beta functions for the two couplings ρ, λ. We use dimensional regularisation to compute the loop integrals, introducing the mass scale μ to give the correct mass dimension to both ρ and λ.

For the special value $\lambda = 3\rho$ the theory possesses a global O(2) symmetry that rotates the two fields among themselves. We will see that this symmetry can be restored in the UV.

a) First draw all the diagrams involved in the one loop correction of the vertices; pay attention to the symmetry factors.

b) Write all the loop integrals. They should all be of the form $V(s)$, $V(t)$ or $V(u)$, where

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - p_3)^2, \qquad u = -(p_1 - p_4)^2 \tag{10.2}$$

are the usual Mandelstam variables for an elastic $2 \to 2$ process and $V(-k^2)$ is the loop integral with four external legs and two internal scalar propagators.

c) Define the counterterms such the tree level result receives no one-loop correction at the point $s = t = u = \mu^2$. Then compute

$$\beta_\lambda(\rho,\lambda) := \mu\,\frac{\partial}{\partial\mu}\lambda, \qquad \beta_\rho(\rho,\lambda) := \mu\,\frac{\partial}{\partial\mu}\rho. \tag{10.3}$$

d) Compute the evolution equation for the ratio λ/ρ in the one-loop approximation — that is, compute the function

$$\beta_{\lambda/\rho}(\lambda,\rho) := \mu\,\frac{\partial}{\partial\mu}\left(\frac{\lambda}{\rho}\right). \tag{10.4}$$

Show that for $1 < \lambda/\rho < 3$ the theory flows in the UV to a point where the O(2) global symmetry is restored.

$\longrightarrow$

10.2. Callan–Symanzik equations in dimensional regularisation

Let us consider a quantum field theory with just a single massless field ϕ (with bare dimension d) and a single dimensionless coupling constant λ, at a regularisation scale μ. We consider the n-point correlation function $G_n(p_k, \lambda, \mu)$, depending on the momenta p_k. Let $\bar{G}_n(p_k, \bar{\lambda}, \mu, \epsilon)$ be the bare correlation function, that depends on the bare coupling $\bar{\lambda}$ and on the regulator ϵ. The renormalised correlation function will be written in terms of the bare one as

$$\bar{G}_n(p_k, \bar{\lambda}, \mu, \epsilon) = N(\bar{\lambda})^n \, G_n(p_k, \lambda(\bar{\lambda}), \mu, \epsilon), \tag{10.5}$$

where G_n is finite, with the divergencies of $\bar{G}_n$ absorbed into the definition of the renormalised coupling $\lambda(\bar{\lambda})$ and of the wave function renormalisation $N(\bar{\lambda})$. Recall that the Callan–Symanzik equation has the general form

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right) G_n(p_k, \lambda, \mu) = 0 \tag{10.6}$$

for a generic n-point correlation function G_n.

a) Assuming that the Callan–Symanzik equation holds for $\bar{G}_n$, show that

$$\bar{\beta}(\bar{\lambda}) = -2\epsilon\bar{\lambda}, \quad \bar{\gamma}(\bar{\lambda}) = 0. \tag{10.7}$$

b) Show that it holds also for G_n, with coefficients

$$\beta(\lambda(\bar{\lambda})) = \bar{\beta}(\bar{\lambda}) \frac{\partial \lambda}{\partial \bar{\lambda}}, \qquad \gamma(\lambda(\bar{\lambda})) = \bar{\gamma}(\bar{\lambda}) + \frac{\bar{\beta}(\bar{\lambda})}{N(\bar{\lambda})} \frac{\partial N}{\partial \bar{\lambda}}. \tag{10.8}$$

c) Now, consider the perturbative expansion of the coupling constant and of the wave function renormalisation which have the general form

$$\lambda(\bar{\lambda}) = \bar{\lambda} + \bar{\lambda}^2 \left(\frac{b_1}{\epsilon} + b_2 + b_3\epsilon + \dots\right) + \dots, \tag{10.9}$$

$$N(\bar{\lambda}) = 1 + \bar{\lambda}\left(\frac{c_1}{\epsilon} + c_2 + c_3\epsilon + \dots\right) + \dots. \tag{10.10}$$

Compute $\beta(\lambda)$ and $\gamma(\lambda)$ and check that they are finite.

d) Now, consider the next perturbative order, that is

$$\lambda(\bar{\lambda}) = \bar{\lambda} + \bar{\lambda}^2 \left(\frac{b_1}{\epsilon} + b_2 + b_3\epsilon + \dots\right) + \bar{\lambda}^3 \left(\frac{a_1}{\epsilon^2} + \frac{a_2}{\epsilon} + a_3 + \dots\right) \dots, \tag{10.11}$$

$$N(\bar{\lambda}) = 1 + \bar{\lambda}\left(\frac{c_1}{\epsilon} + c_2 + c_3\epsilon + \dots\right) + \bar{\lambda}^2 \left(\frac{d_1}{\epsilon^2} + \frac{d_2}{\epsilon} + d_3 + \dots\right) + \dots. \tag{10.12}$$

Compute $\beta(\lambda)$ to order λ^3 and $\gamma(\lambda)$ to order λ^2 and show that their finiteness implies that the coefficients entering the two loop corrections are not independent of the lower order ones.

Why can we say that, whatever renormalisation scheme we choose, the leading order of the beta function for the coupling is universal?

e) Consider higher loop corrections and show schematically what happens to the higher order coefficients. Pay particular attention to the role of the lower perturbative orders.

f) Consider higher-loop corrections and show schematically what happens to the higher-order coefficients. Pay particular attention to the role of the lower perturbative orders.

11.1. Ward–Takahashi identity

The Ward–Takahashi identities in QED are the extension of the conservation of the Noether current for the U(1) global symmetry

$$N_\mu(x) = -q\bar{\psi}(x)\gamma_\mu\psi(x) \tag{11.1}$$

to correlation functions. Let be the Noether current for the global U(1) current of electrodynamics. The Ward–Takahashi identity reads

$$\begin{aligned}&\partial_z^\mu\langle N_\mu(z)\psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)A_{\nu_1}(z_1)\ldots A_{\nu_p}(z_p)\rangle\\ &\quad= -q\langle\psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)A_{\nu_1}(z_1)\ldots A_{\nu_p}(z_p)\rangle\sum_{i=1}^{n}\big(\delta^4(z-y_i)-\delta^4(z-x_i)\big).\end{aligned} \tag{11.2}$$

Define the full propagator for the electron as

$$-iG(k)\delta^4(k-p) := \int d^4y\, d^4z\, \langle\psi(y)\bar{\psi}(z)\rangle e^{ip\cdot z - ik\cdot y}. \tag{11.3}$$

and the electromagnetic vertex function $V^\mu(k,l)$ as

$$\int d^4x\, d^4y\, d^4z\, e^{-i(p\cdot x+k\cdot y-l\cdot z)}\langle N^\mu(x)\psi(y)\bar{\psi}(z)\rangle =: iqG(k)V^\mu(k,l)G(l)\delta^4(p+k-l). \tag{11.4}$$

a) Show that the Ward–Takahashi identity implies that

$$(p-k)_\mu V^\mu(k,p) = iG^{-1}(k) - iG^{-1}(p). \tag{11.5}$$

b) Define the counterterms Z_1 as

$$V^\mu(p,p) := (Z_1)^{-1}\gamma^\mu \tag{11.6}$$

(notice that $V(p,p)$ is the vertex in the limit of zero momentum of the photon) and Z_2 as the residue of $G(p)$ around the pole in m. Show that, with these definitions,

$$Z_1 = Z_2. \tag{11.7}$$

⟶

11.2. The axial anomaly in two-dimensional QED

Consider massless quantum electrodynamics realised in two spacetime dimensions

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{11.8}$$

In this scenario the Lorentz indices are $\mu \in \{0, 1\}$ and the Dirac matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}. \tag{11.9}$$

Here we choose them to be 2×2 matrices

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{11.10}$$

The two components of the spinor ψ satisfy the Dirac equation separately. The matrix γ^3 is the two-dimensional analog of γ^5 in four dimensions

$$\gamma^3 = \tfrac{1}{2}\varepsilon_{\mu\nu}\gamma^\mu\gamma^\nu = \gamma^0\gamma^1, \tag{11.11}$$

where $\varepsilon^{\mu\nu}$ is the totally anti-symmetric tensor in two dimensions. It satisfies

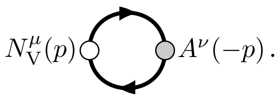
$$\gamma^\mu\gamma^3 = \varepsilon^{\mu\nu}\gamma_\nu. \tag{11.12}$$

In this exercise we consider the spinor field ψ to be quantised and the electromagnetic field A_μ to take some fixed classical configuration.

a) Convince yourself using the equations of motion that this leads to the separate conservation of the vector and axial current

$$J^\mu_{\mathrm{V}} = \bar{\psi}\gamma^\mu\psi, \qquad J^\mu_{\mathrm{A}} = \bar{\psi}\gamma^3\gamma^\mu\psi. \tag{11.13}$$

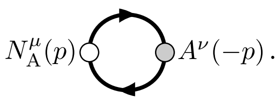
b) Calculate the one-loop correction to $\langle N^\mu_{\mathrm{V}}(p)\rangle$ using dimensional regularisation to linear order in the gauge field A. In other words, compute the graph



(11.14)

Check that the current is still conserved at the one-loop level. How do you interpret your result in terms of renormalisation?

c) Show that the axial current is no longer conserved at the quantum level. To this end compute $\langle N^\mu_{\mathrm{A}}(p)\rangle$ at one loop and at linear order in A



(11.15)

How could this happen? At which point of your calculation did you introduce this quantum anomaly?

Hint: Use (11.12) to relate the axial to the vector current. Do not use (11.10) and (11.11)!

d) Can you spot the anomaly in position space, i.e. in $\langle \partial{\cdot}N_{\mathrm{V}}(x)\rangle$ vs. $\langle \partial{\cdot}N_{\mathrm{A}}(x)\rangle$?

Hint: No regularisation is required here.

12.1. The Higgs mechanism in the standard model

In this exercise we review the Higgs mechanism in the standard model. Let us regard first of all the mass terms of matter fields. Consider the first lepton generation, i.e. the electron and its neutrino. The electroweak gauge group is $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ (isospin and hypercharge). The left- and right-handed leptons appear in different structures in the Lagrangian. The left-handed fermions form a doublet with respect to $\mathrm{SU}(2)_I$

$$L_\mathrm{L} := \begin{pmatrix} \nu_\mathrm{L} \\ e_\mathrm{L} \end{pmatrix}, \qquad L'_\mathrm{L} = \exp(i\theta^a I_a) L_\mathrm{L}, \tag{12.1}$$

where the $I_a = \frac{1}{2}\sigma_a$ are the generators of $\mathrm{SU}(2)_I$ in the fundamental representation (σ_a are the Pauli matrices). The right-handed electron forms a singlet under $\mathrm{SU}(2)_I$.

$$e'_\mathrm{R} = e_\mathrm{R}. \tag{12.2}$$

To first approximation we will not need a right-handed neutrinos.
Under $\mathrm{U}(1)_Y$ the fields transform as

$$L'_\mathrm{L} = e^{-i\theta/2} L_\mathrm{L}, \qquad e'_\mathrm{R} = e^{-i\theta} e_\mathrm{R}. \tag{12.3}$$

a) Why is a mass-term of the form

$$\mathcal{L}_\mathrm{mass} := -m_e(\bar{e}_\mathrm{L} e_\mathrm{R} + \mathrm{h.c.}) \tag{12.4}$$

not allowed?

b) In order to assign a mass to our fields we introduce a new scalar field H. This field couples to the electron and neutrino as

$$\mathcal{L}_\mathrm{Yukawa} := y \bar{L}_\mathrm{L} H L_\mathrm{R} + \mathrm{h.c.}, \tag{12.5}$$

where y is the coupling constant. How does the field H have to transform to maintain gauge invariance?

c) Now, that you have shown that H transforms as a doublet under $\mathrm{SU}(2)_I$, find the according gauge transformation U to bring H into the following form

$$UH = U \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} 0 \\ H_\mathrm{r} \end{pmatrix}, \tag{12.6}$$

where H_r is real. This specific gauge is usually referred to as unitary gauge.

⟶

d) We add a potential for the scalar field

$$\mathcal{L}_{\text{pot}} = \mu^2(H^\dagger H) - \tfrac{1}{2}\lambda(H^\dagger H)^2. \tag{12.7}$$

The minimum of the potential is the vacuum expectation value of the field $v := \langle H_\text{r} \rangle$. Find the minima of the potential and show that for $\mu^2 > 0$ the minimum is at $v \neq 0$.

e) We choose $\mu^2 > 0$ and redefine H_r in order to make the physical degrees of freedom more obvious $H_\text{r} = (v + \eta)/\sqrt{2}$. η is now the physical Higgs boson field. Write down (12.5) in terms of the Higgs field and the vacuum expectation value. You will see that the fermions acquire a mass term. Write down the fermion mass in terms of y and v. What is the coupling of the Higgs boson to the fermions in terms of m_e and v?

The standard model Lagrangian density also contains a kinetic term for the Higgs doublet

$$\mathcal{L}_{\text{kin}} := -(D_\mu H)^\dagger (D^\mu H). \tag{12.8}$$

The covariant derivative acting on the Higgs doublet involves the gauge fields for the $\mathrm{SU}(2)_I$ and $\mathrm{U}(1)_Y$ symmetry W^a_μ and B_μ, respectively

$$D_\mu = \partial_\mu - igI_a W^a_\mu - ig'YB_\mu. \tag{12.9}$$

In the following you should try to find the masses and the couplings to the Higgs boson of the physical standard model fields.

f) Diagonalise the quadratic term by introducing the physical fields

$$W^+_\mu = (W^-_\mu)^\dagger = \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu), \quad Z^0_\mu = \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2 + g'^2}}, \quad A_\mu = \frac{gB_\mu + g'W^3_\mu}{\sqrt{g^2 + g'^2}}. \tag{12.10}$$

The hypercharge of the Higgs boson is $Y = \frac{1}{2}$. Now you can read off the masses and couplings of the theory by comparing your results to

$$\begin{aligned}(D_\mu H)^\dagger (D^\mu H) = {}& \tfrac{1}{2}(\partial_\mu \eta)^2 + m_W^2 W^+_\mu W^{-,\mu} + \tfrac{1}{2} m_Z^2 Z^{0,\mu} Z^0_\mu \\ &+ \rho_W \eta W^+_\mu W^{-,\mu} + \tfrac{1}{2}\lambda_W \eta^2 W^+_\mu W^{-,\mu} \\ &+ \tfrac{1}{2}\rho_Z \eta Z^{0,\mu} Z^0_\mu + \tfrac{1}{4}\lambda_Z \eta^2 Z^{0,\mu} Z^0_\mu.\end{aligned} \tag{12.11}$$

g) The original doublet H included 4 degrees of freedom. What did the introduction of the vacuum expectation value v do to them? Which symmetry was broken? Where are the degrees of freedom hidden after spontaneous symmetry breaking? Why did the photon field A_μ not acquire a mass term? Why does it couple to the vector currents, but not to the axial currents?