Exercise 1. ***Rarefaction Waves***

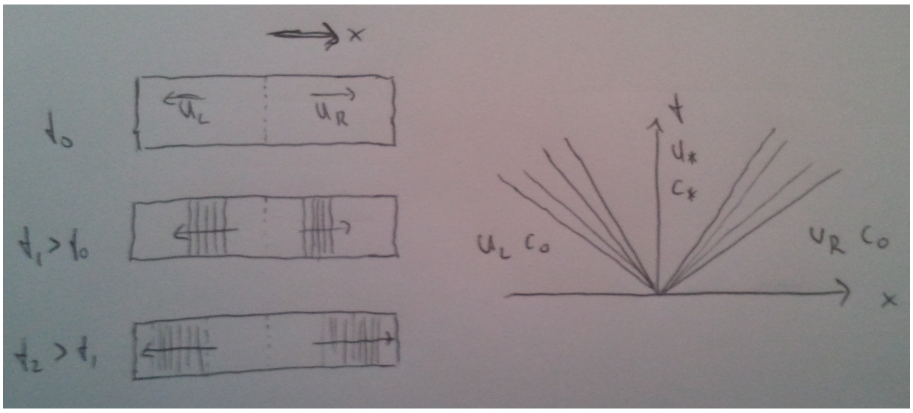

Figure 1: Some mechanism imposes velocities u_R and u_L in two parts of an initially quiescent fluid. This triggers rarefaction waves/fans, which spread out over time. Before the rarefaction waves hit, the fluid state is (u_L, c_0) and (u_R, c_0), respectively. Once the wave passes, the fluid has state $(u_\star, c_\star)$.

In this exercise we consider a one-dimensional fluid initially at rest with constant sound speed c_0 everywhere. Some mechanism sets the flow in motion such that for $x > 0$, the fluid moves with velocity $u_R(x,t)$ (right), and for $x < 0$, it moves with velocity $u_L(x,t)$ (left). The leads to a left- and a right-running rarefaction fan/wave moving with velocity $u - c$ and $u + c$ launched from $x = 0$ at $t = 0$; cf. figure 1. We are interested in the state of the fluid between the two rarefaction waves – the $\star$-state characterized by the sound speed $c_\star$ and velocity $u_\star$.

(a) Firstly, we have to find a way to relate some state of the fluid to another state. For a barotropic equation of state $p(\rho) = C\rho^\gamma$, show how density ρ and pressure p of two states (p, ρ) and (p_0, ρ_0) in an isentropic (reversibly adiabatic) flow are related.

(b) Write down the one-dimensional equation for mass and momentum conservation, use density ρ, pressure p, position x, velocity u, and time t. Show that they can be rewritten as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{1}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0. \tag{2}$$

(c) We will now rewrite the equations for mass and momentum conservation in terms of some variable $\alpha^\pm$. If we can show that $\alpha^\pm$ is conserved across rarefaction waves, we can relate the state of the fluid outside of the rarefaction wave to the $\star$-state between the waves.

Using the sound speed

$$c_s^2 = \frac{\gamma p}{\rho}, \tag{3}$$

show that (2) can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x} = 0. \tag{4}$$

Hint: You may want to figure out what $\partial p/\partial x$ can be rewritten to.

(d) Show that

$$\frac{1}{c_s}\frac{\partial c_s}{\partial x} = \frac{1}{\rho}\frac{\partial \rho}{\partial x}\left(\frac{\gamma-1}{2}\right), \quad \text{and} \quad \frac{1}{c_s}\frac{\partial c_s}{\partial t} = \frac{1}{\rho}\frac{\partial \rho}{\partial t}\left(\frac{\gamma-1}{2}\right), \tag{5}$$

and demonstrate that (1) and (4) can be written as

$$\frac{\partial}{\partial t}\left(\frac{2}{\gamma-1}\,c_s\right) + u\,\frac{\partial}{\partial x}\left(\frac{2}{\gamma-1}\,c_s\right) + c_s\,\frac{\partial u}{\partial x} = 0, \tag{6}$$

and

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + c_s\,\frac{\partial}{\partial x}\left(\frac{2}{\gamma-1}\,c_s\right) = 0. \tag{7}$$

Hint: You might need to reuse your previous result for $\partial p/\partial x$, and calculate $\partial p/\partial t$.

(e) We are now in a position to define the variables $\alpha^{\pm}$. Let

$$\alpha^{+} = u + \frac{2}{\gamma-1}\,c_s, \qquad \text{and} \qquad \alpha^{-} = u - \frac{2}{\gamma-1}\,c_s, \tag{8}$$

and use the mass and momentum equations (6) and (7) to derive

$$\frac{\partial \alpha^{+}}{\partial t} + (u+c_s)\,\frac{\partial \alpha^{+}}{\partial x} = 0, \qquad \text{and} \qquad \frac{\partial \alpha^{-}}{\partial t} + (u-c_s)\,\frac{\partial \alpha^{-}}{\partial x} = 0. \tag{9}$$

(f) Sketch the left- and right-running rarefaction waves moving with velocity $u-c_s$ and $u+c_s$ in a (x,t) diagram. In the same diagram, sketch α^{-} and α^{+}. Note that α^{+} crosses the left running rarefaction wave, while α^{-} crosses the right running rarefaction wave. Why?

(g) We can only proceed if we can show that $\alpha^{\pm}$ is conserved across the rarefaction waves. So, determine whether $\alpha^{\pm} = \alpha^{\pm}(x(t),t)$ are conserved.

(h) Given the previous, write down equations relating the left state (u_L, c_0), the right state (u_R, c_0), and the $\star$-state $(u_\star, c_\star)$ between the two rarefaction waves. Investigate and interpret $(u_\star, c_\star)$ in the following cases.

   (i) What happens for $u_R = -u_L$?

   (ii) What happens for $u_R = u_L$?

   (iii) Under what conditions is $c_\star \leq 0$.

Exercise 2. ***Sedov-Taylor Blast Wave***

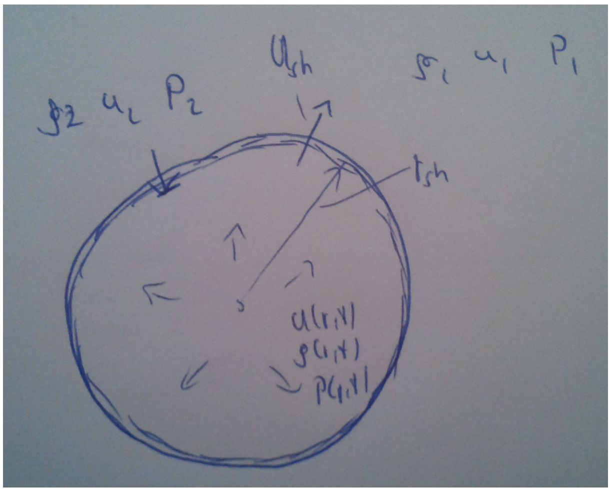

Figure 2: A blast wave of size $r_{\rm sh}$ is expanding into a region characterized by ρ_1, u_1, P_1 with velocity $U_{\rm sh}$. Directly behind the shock, conditions are ρ_2, u_2, and P_2. Further behind the shock, conditions are $u(r,t)$, $\rho(r,t)$, and $P(r,t)$.

Explosions such as those of supernovae of atomic bombs generate strong blast waves. In this exercise, we consider a simple blast wave generated by injecting a finite amount of energy E into a singular point in a fluid characterized by a uniform density ρ_1, as well as velocity u_1 and pressure P_1. Once injected, a blast wave will propagate in a spherically symmetric fashion outwards along the radial coordinate r. In this exercise, we derive the velocity field $u(r,t)$, pressure $P(r,t)$, and density $\rho(r,t)$ far behind the shock, cf. figure 2.

(a) Write the Euler equations for the conservation of mass, momentum, and energy in the absence of external forces. Assuming spherical symmetry, derive the Euler equations in spherical coordinates.

To proceed later, make sure to write out the total energy $E = E(\rho, \epsilon, \vec{u})$, where ϵ is the specific internal energy, and $\vec{u}$ the velocity vector. To obtain ϵ, assume an ideal gas.

(b) We now want to non-dimensionalize the spherical Euler equations obtained in the previous task. Since the problem is spherically symmetric, the only relevant coordinate in the problem is the radial distance r (and of course the time t).

Given the energy injected E, the undisturbed density ρ_1, and time t, perform dimensional analysis to recover the non-dimensional radius $\xi = \xi(r, \rho_1, E, t)$. Also write down ξ_0 at the shock front $r_{\rm sh}$.

(c) Let us introduce the non-dimensional density $\alpha(\xi)$, velocity $v(\xi)$, and pressure $p(\xi)$. Use dimensional arguments to show that

$$\rho(r,t) = \rho_2\, \alpha(\xi), \tag{10}$$

$$u(r,t) = \frac{r}{t}\, v(\xi), \tag{11}$$

$$P(r,t) = \rho_1 \frac{r^2}{t^2}\, p(\xi). \tag{12}$$

(d) Given the appropriate jump conditions across the shock (which you will learn about in a later lecture), we introduce numerical convenience factors in (10), (11), and (12). Using

$$
\begin{aligned}
\rho(r,t) &= \rho_2\,\alpha(\xi), && (13)\\
u(r,t) &= \left(\frac{4}{5(\gamma+1)}\right)\frac{r}{t}\,v(\xi), && (14)\\
P(r,t) &= \left(\frac{8}{25(\gamma+1)}\right)\rho_1\frac{r^2}{t^2}\,v(\xi), && (15)
\end{aligned}
$$

rewrite the spherical Euler equations to obtain the non-dimensional form of the mass, momentum, and energy equation as

$$-\xi\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}+\frac{2}{\gamma+1}\left[3\alpha v+\xi\frac{\mathrm{d}}{\mathrm{d}\xi}(\alpha v)\right]=0, \tag{16}$$

$$-v-\frac{2}{5}\xi\frac{\mathrm{d}v}{\mathrm{d}\xi}+\frac{4}{5(\gamma+1)}\left(v^2+v\xi\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)=-\frac{2}{5}\left(\frac{\gamma-1}{\gamma+1}\right)\frac{1}{\alpha}\left(2p+\xi\frac{\mathrm{d}p}{\mathrm{d}\xi}\right), \tag{17}$$

and

$$-2(p+\alpha v^2)-\frac{2}{5}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}(p+\alpha v^2)+\frac{4}{5(\gamma+1)}\left\{5v(\gamma p+\alpha v^2)+\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\left[v\left(\gamma p+\alpha v^2\right)\right]\right\}=0. \tag{18}$$

To do this properly, you might need the following relations

$$\frac{\rho_2}{\rho_1}=\frac{\gamma+1}{\gamma-1},\qquad \frac{\partial}{\partial t}=-\frac{2}{5}\frac{\xi}{t}\frac{\mathrm{d}}{\mathrm{d}\xi},\qquad \frac{\partial}{\partial r}=\frac{\xi}{r}\frac{\mathrm{d}}{\mathrm{d}\xi}. \tag{19}$$

(e) We are now left with three equations (16), (17), and (18), and three unknown derivatives $\mathrm{d}\alpha/\mathrm{d}\xi$, $\mathrm{d}v/\mathrm{d}\xi$, and $\mathrm{d}p/\mathrm{d}\xi$, which we can solve for. Knowing these, we can then pick a numerical integration scheme of choice to recover $\alpha(\xi)$, $v(\xi)$, and $p(\xi)$ by integrating from $\xi=\xi_0$ to $\xi=0$.

However, we do not actually know the shock position ξ_0, and therefore have no idea where to start the integration. However, we can guess a value for ξ_0, integrate (16), (17), and (18) to obtain $\alpha(\xi)$, $v(\xi)$, and $p(\xi)$, and then check if the result is reasonable.

Additionally, we must rescale $\alpha(\xi)$, $v(\xi)$, and $p(\xi)$ to their relevant dimensional counterparts. To this end, note that

$$\frac{\rho(r,t)}{\rho_2}=\alpha(\xi),\qquad \frac{u(r,t)}{u_2}=\frac{r}{r_{\mathrm{sh}}}v(\xi),\qquad \frac{P(r,t)}{P_2}=\left(\frac{r}{r_{\mathrm{sh}}}\right)^2p(\xi), \tag{20}$$

where you should have obtained r/r_{sh} in (b).

Implement a numerical integration scheme of your choice, use it to calculate $\alpha(\xi)$, $v(\xi)$, and $p(\xi)$. Then plot ρ/ρ_2, u/u_2, and P/P_2 from some ξ_0 of your choice to $\xi=0$. Use $\gamma=5/3$.

Hint: Due to the numerical factors introduced in (13), (14), *and* (13), *we have nice initial conditions for the integration –* $\alpha(\xi_0)=v(\xi_0)=p(\xi_0)=1$.

Hint: The differential equations (16)*,* (17)*, and* (18) *are rather unwieldy.*[1] *Usage of a CAS such as Maple or Mathematica to solve for the derivatives is probably not a bad idea unless you enjoy tracking down that one elusive minus sign. Hand in any notebooks though!*

Hint: You will need at least a second order accurate integration scheme (so the standard Euler method will not cut it) with ≥ 4096 *sampling points. Be careful as you approach* $\xi = 0$.

(f) While your result might look reasonable, we have to check whether we actually have chosen ξ_0 as the actual shock position. In fact, energy conservation gives the constraint

$$\frac{32}{25(\gamma^2-1)} \int_0^{\xi_0} \left(p(\xi) + \alpha(\xi) v(\xi)^2\right) \xi^4 \, \mathrm{d}\xi = 1. \tag{21}$$

Numerically evaluate (21) for your solution of $\alpha(\xi)$, $v(\xi)$, and $p(\xi)$. If (21) is not fulfilled, iterate on your guess of ξ_0 until it is (to some accuracy).

[1] Sedov actually solved this system of equations analytically.