Exercise 1. ***Velocity fields and dynamics of vortex lines***

(i) Calculate the velocity field $\vec{v}(\vec{r})$ induced by a line vortex given by

$$\vec{\omega}(\vec{r}) = \Omega\ \delta(x - x_0)\ \delta(y - y_0)\ \vec{e}_z. \tag{1}$$

(ii) Consider two line vortices A and B with strength Ω_A and Ω_B, which at $t = 0$ are at position $(x_{A,0}, y_{A,0})$ and $(x_{B,0}, y_{B,0})$ with $\Omega_A = \Omega_B$ and $x_{A,0} = -x_{B,0}$ and $y_{A,0} = -y_{B,0}$). How do these two vortices move?

(iii) Consider two line vortices A and B with strength Ω_A and Ω_B, which at $t = 0$ are at position $(x_{A,0}, y_{A,0})$ and $(x_{B,0}, y_{B,0})$ with $\Omega_A = -\Omega_B > 0$ and $x_{A,0} = x_{B,0} = x_0 > 0$ and $y_{A,0} = -y_{B,0} = -d$. How do these two vortices move?

(iv) Consider the two line vortices from part (iii) in the half space $x > 0$ bounded by a solid wall at $x = 0$ (boundary condition $\vec{v} \cdot \vec{n}|_{x=0} = 0$). How do these two vortices move? *Hint: Analogous to mirror charges in electrostatics, introduce mirror line vortices to satisfy the boundary conditions.*

Exercise 2. ***Dynamics of a vortex ring***

Explain qualitatively how the vortex ring,

$$\vec{\omega}(\vec{r}) = \Omega\ \delta(y^2 + z^2 - R^2)\ \delta(x - x_0) \begin{pmatrix} 0 \\ z/\sqrt{y^2+z^2} \\ -y/\sqrt{y^2+z^2} \end{pmatrix} \tag{2}$$

moves

(i) in free space,

(ii) in the half plane $x > 0$,

using the knowledge of exercise 1(iii) and 1(iv).

Exercise 3. ***Velocity field of a vortex ring***

Show that the velocity field induced by the vortex ring

$$\vec{\omega}(\vec{r}) = \Omega\ \delta(x^2 + y^2 - R^2)\ \delta(z) \begin{pmatrix} -y/\sqrt{x^2+y^2} \\ x/\sqrt{x^2+y^2} \\ 0 \end{pmatrix} \tag{3}$$

looks like a dipole field as seen from large distances. *Hint: Calculate*

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r', \tag{4}$$

with $\vec{v}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$ *for* $|r| \gg R$. *Compare to the magnetic field of a current loop.*

Exercise 4. ***Shallow water***

In the lecture the shallow water equations in one space dimension have been derived. We look at the case of two space dimensions. Let the velocity vector be defined by $w := (u, v)$. The shallow water equations are given in conservative form by

$$\frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0, \tag{5}$$

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(hu^2 + \tfrac{1}{2}gh^2) + \frac{\partial}{\partial y}(huv) = 0, \tag{6}$$

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial y}(hv^2 + \tfrac{1}{2}gh^2) + \frac{\partial}{\partial x}(huv) = 0, \tag{7}$$

representing conservation of mass (5) and momentum (6), (7). Here g is the gravitational acceleration and h the height of the water layer.

(i) On the basis of (5), (6) and (7) derive the Lagrangian form of the shallow water equations using the Lagrangian derivative

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + w \cdot \nabla, \tag{8}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$. Show that the equation for the conservation of momentum is equivalent to the one of compressible fluid flow if we look at a stream line lying in the x-y-plane (i.e. the streamline being covered by water of height h) and assume $\rho =$ const. What role does the height h play in the compressible fluid equations?

(ii) On the basis of the Lagrangian form of the conservation of momentum derive the equation which is analogous to the Bernoulli equation of compressible fluid flow. Use the assumption of stationary flow.

(iii) We now apply the equations to stationary flow in a channel along the x-direction. Let the velocity point always in the x-direction, i.e. $w = (u, 0)$ (an approximation valid for channels with moderate changes of width) and let the width of the channel be given as a function of x by $b(x)$. On the basis of the Lagrangian form of the conservation of mass, show that

$$h(x)u(x)b(x) = M, \tag{9}$$

where M is the (constant) mass flow through the channel. The product $h(x)b(x)$ being the area of the cross section of the channel, validation of the result is immediate.

(iv) In the lecture the Hugoniot theorem for compressible stationary flow in one space dimension was derived. Derive the analogous equation for stationary channel flow using the Lagrangian form of the momentum conservation as derived in (i) together with (9). Interpret the result.