Exercise 1. ***Streamlines and Trajectories***

We consider three different flows. For each flow, determine (i) whether it is compressible, (ii) the streamlines, and (iii) the trajectories. Sketch streamlines and trajectories.

(a) Before moving on, briefly describe the difference between streamlines and trajectories.

(b) The stationary flow

$$\vec{v} = (v_x, v_y, v_z) = (x, y, 0). \tag{1}$$

Sketch streamlines and trajectories originating at $(x_0, y_0) = (1, 1)$.

(c) The non-stationary flow

$$\vec{v} = (v_x, v_y, v_z) = \left(\frac{x}{1+t}, \frac{y}{1+2t}, 0 \right). \tag{2}$$

Sketch streamlines and trajectories originating at $(x_0, y_0) = (1, 1)$ for time $t = t_0 = 0$.

(d) The Rankine vortex

$$\vec{v} = (v_r, v_\theta, v_z) = \left(0, \frac{\Gamma}{r}, 0 \right), \tag{3}$$

where Γ is a constant (the circulation). You should find that particles move on circular trajectories. Is this realistic for an actual vortex in nature? Why?

Exercise 2. ***Ideal Gas Equation of State – Compressibility and Sound Speed***

(a) Consider the ideal gas equations of state

$$p = (\gamma - 1)\rho\epsilon, \tag{4}$$

where p is the pressure, γ the ratio of specific heats, ρ the density, and $\epsilon = e/\rho$ the internal energy per unit mass. Show that (4) is the same as the more familiar equation of state for an ideal gas

$$pV = Nk_BT, \tag{5}$$

where N is the number of particles in the volume V, p the pressure, k_B the Boltzmann constant, and T the temperature.

(b) If we consider a fixed number of particles N within a volume V, the change in occupied volume $\mathrm{d}V$ as pressure and temperature are varied by $\mathrm{d}p$ and $\mathrm{d}T$ is

$$\mathrm{d}V = \left.\frac{\partial V}{\partial p}\right|_{T=\mathrm{const}} \mathrm{d}p + \left.\frac{\partial V}{\partial T}\right|_{p=\mathrm{const}} \mathrm{d}T. \tag{6}$$

Dividing by $(-V)$, and noting that $\rho \mathrm{d}V + V\mathrm{d}\rho = 0$, we rewrite (6) and define the compressibility coefficient (at constant temperature) β_T and the thermal expansion coefficient α as

$$\frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}V}{V} = \underbrace{-\frac{1}{V}\left.\frac{\partial V}{\partial p}\right|_{T=\mathrm{const}}}_{\beta_T} \mathrm{d}p - \underbrace{\frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_{p=\mathrm{const}}}_{\alpha} \mathrm{d}T. \tag{7}$$

For an ideal gas, compute β_T and α.

(c) We now define an adiabatic compression coefficient at constant entropy

$$\beta_S = -\left.\frac{1}{V}\frac{\partial V}{\partial p}\right|_{S=\mathrm{const}}. \tag{8}$$

For adiabatic processes, no heat transfer takes place such that the change in internal energy is just $\mathrm{d}e = -p\,\mathrm{d}V = C_v\,\mathrm{d}T$. For an ideal gas, show that

$$\frac{\beta_S}{\beta_T} = \frac{1}{\gamma}. \tag{9}$$

(d) Let us introduce the bulk moduli B_i as the inverse of the compressibilities β_i, i.e.

$$B_T = \frac{1}{\beta_T}, \qquad B_S = \frac{1}{\beta_S}. \tag{10}$$

Let us further define the sound speed

$$c_s^2 = \frac{\partial p}{\partial \rho}. \tag{11}$$

Derive the expressions for the isothermal ($\mathrm{d}T = 0$) and adiabatic sound speeds. Why are they different? What fluid property determines the actual sound speed the medium?

Exercise 3. ***Stiffened Equation of State – Compressibility and Sound Speed***

Consider the expression for the Helmholtz free energy

$$A(\rho, T) = c_V T \left(1 - \ln\left(\frac{T}{T_0}\right) + (\gamma - 1)\ln\left(\frac{\rho}{\rho_0}\right)\right) - s_0 T + \frac{p_\infty}{\rho} + \epsilon_*, \tag{12}$$

where c_V, γ, p_∞, ϵ_* are constants specific to the continuum under consideration. From A, we can obtain entropy s, specific internal energy ϵ, and pressure p as

$$s(\rho, T) = -\left.\frac{\partial A}{\partial T}\right|_{\rho=\text{const}}, \qquad \epsilon(\rho, T) = A + T\,s, \qquad p(\rho, T) = \rho^2 \left.\frac{\partial A}{\partial \rho}\right|_{T=\text{const}}. \tag{13}$$

Also recall specific heat capacities and adiabatic sound speed as

$$c_V = \left.\frac{\partial \epsilon}{\partial T}\right|_{V=\text{const}}, \qquad c_p = T\left.\frac{\partial s}{\partial T}\right|_{p=\text{const}}, \qquad c_s^2\big|_{\text{adiabatic}} = \left.\frac{\partial p}{\partial \rho}\right|_{s=\text{const}}. \tag{14}$$

(a) From $A(\rho, T)$, derive the stiffened equation of state

$$p(\rho, \epsilon) = (\gamma - 1)\rho(\epsilon - \epsilon_*) - \gamma p_\infty. \tag{15}$$

How does it compare to the equation of state for an ideal gas?

(b) Demonstrate that c_V is indeed the specific heat capacity at constant volume, c_p the specific heat capacity at constant pressure, and γ the ratio of specific heats.

(c) For the stiffened equation of state, compute β_T and α as previously done for the ideal gas equation of state, cf. (7). Compare!

(d) Show that the adiabatic sound speed is

$$c_s^2 = \gamma \frac{p + p_\infty}{\rho}. \tag{16}$$

With this in mind, interpret p_∞ and ϵ_* in (12) and (15) and compare to an ideal gas.

(e) Finally, consider the Mie-Grüneisen equation of state discussed in the lecture

$$p - p_c = \Gamma \rho(\epsilon - \epsilon_c), \tag{17}$$

with sound speed

$$c_s^2 = p_c' + (\Gamma + 1)\,\frac{p - p_c}{\rho}, \tag{18}$$

and discuss how it compares to the stiffened equation of state!