Exercise 1. ***Waves in an inhomogeneous medium***

In this exercise we want to study the behaviour of p-waves in an elastic medium with a spatially dependent mass density.

a) Consider two elastic media (1 and 2) which are separated by a plane at $x = 0$, see Fig. 1. The characteristics $\lambda_{1,2}, \mu_{1,2}, \rho_{1,2}$ of the two media are constant but differ from each other.

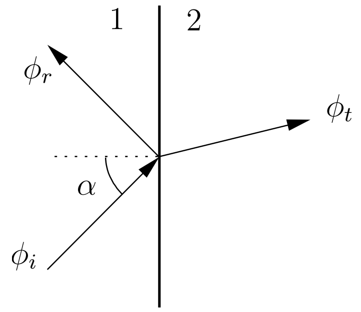

Figure 1: Two media separated by a plane.

The solutions for the scalar function ϕ (with $\vec{u} = \nabla\phi$) are plane waves on both sides but with different wave vectors $|k_1| \neq |k_2|$. An elastic wave ϕ_i hits the wall under an angle α and splits into two waves, a reflected wave ϕ_r and a transmitted wave ϕ_t. The goal is to calculate the ratio t, r of the amplitudes for $\alpha = 0$,

$$t = \frac{A_t}{A_i} \ , \quad r = \frac{A_r}{A_i}, \tag{1}$$

where

$$\begin{aligned} \phi_i &= A_i e^{i(\omega t - k_1 x)} \\ \phi_r &= A_r e^{i(\omega t + k_1 x)} \\ \phi_t &= A_t e^{i(\omega t - k_2 x)}. \end{aligned}$$

(i) Show that the continuity equations for ϕ at the interface $x = 0$ are given by

$$\partial_x \phi_1|_{x=0} = \partial_x \phi_2|_{x=0} \ , \quad c_1^2 \rho_1 \ \partial_x^2 \phi_1|_{x=0} = c_2^2 \rho_2 \ \partial_x^2 \phi_2|_{x=0}. \tag{2}$$

Hint: The continuity conditions for ϕ at the interfaces are determined by the continuity condition on the displacement $u(x,t)$ and the continuity condition on the normal force $\vec{T} = \bar{\bar{\sigma}} \vec{e}_x$.

(ii) Find t and r and use the abbreviation $\mathcal{C} = \rho c$!

b) Next we focus on the ratio between the amplitudes of the wave $u(x,t)$ which is given by

$$\bar{t} = \frac{u_t}{u_i} = \frac{c_1}{c_2} t = \frac{2\mathcal{C}_1}{\mathcal{C}_1 + \mathcal{C}_2} \tag{3}$$

and try to generalize the result to the case of a material with a smoothly varying mass density $\rho(x)$. In order to do so, start from the expression for $\bar{t}$ and derive a differential equation for $I(x)$, the amplitude of the wave $u(x,t)$. Solve the differential equation!

Hint: The ratio $\bar{t}$ describes now how the amplitudes at infinitesimaly close points are connected, i.e.

$$\bar{t} = \frac{I(x+dx)}{I(x)} = \frac{I(x) + \partial_x I(x)\ dx}{I(x)} = 1 + \frac{\partial_x I(x)}{I(x)} dx. \tag{4}$$

Similarly, we have

$$\mathcal{C}_1 \to \mathcal{C}(x)\ , \quad \mathcal{C}_2 \to \mathcal{C}(x+dx) = \mathcal{C}(x) + \partial_x \mathcal{C}(x) dx. \tag{5}$$

c) Unfortunately we were very naive and the result from (b) is wrong from a general point of view. Why? In case one needs the exact result, what would you have to do technically? For the case that $1/c(x) = a - bx$ it turns out that the result from (b) is in very good agreement with the exact results for $\omega \gg \omega_{crit}$ whereas for $\omega < \omega_{crit}$ we have a large descrepancy and the exact result shows even oscillations in ω. Try to find out what is happening! Estimate ω_{crit} out of simple values as the characteristic lengh l (e.g. the distance between extremal points in 1/c(x)), the wave length $\lambda = 2\pi/k$ and the average sound velocity c_0.

Exercise 2. ***Love waves***

Love waves belongs together with the Rayleigh waves to the so-called surface waves. In this exercise we would like to calculate the dispersion relation and afterwards we study the behaviour of a Gaussian wave packet moving in a dispersive material.

a) The dispersion relation of Love waves is defined by the equation

$$\tan(k_1 H) = \frac{\mu_2}{\mu_1} \frac{k_2}{k_1} \tag{6}$$

where k_1, k_2 are given by

$$k_1^2 = \frac{\omega^2}{c_1^2} - k^2\ , \quad k_2^2 = k^2 - \frac{\omega^2}{c_2^2}. \tag{7}$$

For each ω there exists several solutions. However, in this exercise we concentrate on the 0-th solution. In the lecture, we have seen that for the 0-th solution we have

$$\frac{\omega}{k} = c \approx c_1 \ \Rightarrow k_1 H \ll 1. \tag{8}$$

Calculate the dispersion relation $\omega(k)$ of the 0-th mode! Expand it up to third order in k and show that $\omega/k \neq \partial\omega/\partial k$, i.e. the material is dispersive.

Hint: For small x the tangent behaves like $\tan x \approx x$. Furthermore, we introduce the dimensionless values s, q and α as

$$s = \frac{c}{c_1}\ , \quad q = \frac{c_2}{c_1}\ , \quad \alpha = Hk\frac{\mu_2}{\mu_1}. \tag{9}$$

b) Let us consider a general dispersive material with $\omega = \omega_k$. In order to see the influence of a dispersive material on an elastic wave, we want to study a Gaussian wave packet moving through a medium. The wave is given by

$$u(x,t) = \int \frac{dk}{2\pi} A(k) e^{i(\omega_k t - kx)}. \tag{10}$$

where

$$A(k) = \sqrt{2\pi} L \; e^{-\frac{1}{2}L^2(k-k_0)^2}. \tag{11}$$

(i) Calculate $u(x,t)$ for a non-dispersive material, i.e. $\omega = ck$! What is the velocity of the Gaussian? What about the width?

(ii) Calculate $u(x,t)$ for a dispersive material! What is the velocity of the Gaussian? What about the width? Interpret your results!