Exercise 1. ***Clapeyron theorem***

In this task we assume that there are is no body force[1]

$$\overline{\nabla} \cdot \overline{\overline{\sigma}} = \overline{0}. \tag{1}$$

Under the assumption of linear elasticity, the elastic potential is given by the quadratic form

$$\phi = \frac{1}{2} \operatorname{Tr} \left\{ \overline{\overline{\epsilon}}\, \overline{\overline{\overline{\overline{\Gamma}}}}\, \overline{\overline{\epsilon}} \right\} = \frac{1}{2} \operatorname{Tr} \left\{ \overline{\overline{\sigma}}\, \overline{\overline{\epsilon}} \right\}, \tag{2}$$

thus the elastic energy for an arbitrary volume V is given by the integral

$$E = \frac{1}{2} \int_V \mathrm{d}V \ \operatorname{Tr} \left\{ \overline{\overline{\sigma}}\, \overline{\overline{\epsilon}} \right\}. \tag{3}$$

Prove the Clapeyron theorem

$$E = \frac{1}{2} \int_V \mathrm{d}V \ \operatorname{Tr} \left\{ \overline{\overline{\sigma}}\, \overline{\overline{\epsilon}} \right\} = \frac{1}{2} \int_{\partial V} \mathrm{d}S \, \overline{T} \cdot \overline{u}. \tag{4}$$

Hint: use the divergence theorem

$$\int_V \mathrm{d}V \, \overline{\nabla} \cdot \overline{F} = \int_{\partial V} \mathrm{d}S \, \overline{F} \cdot \overline{n}. \tag{5}$$

Exercise 2. ***Building a house***

Assume we want to build a house with a circular ground plan of radius r_0. The house will be built on a massive base plate – so we may do the approximation that the pressure from above is uniform. Furthermore, to keep it simple we assume no friction between the base plate and the ground, thus the stress vector at ground level ($z = 0$) is

$$vcT(x, y, z = 0) = \begin{cases} p_0 \overline{e}_z & \text{for } \sqrt{x^2 + y^2} \leq r_0, \\ 0 & \text{for } \sqrt{x^2 + y^2} > r_0. \end{cases} \tag{6}$$

Use the solution by Boussinesq[2] to examine the possibility of breaking the rock under the house using the von Mises yield criterion. Numerical examination show that the most critical point (e.g. with highest J_2^*) is at the symmetry axis ($r = 0$), which does greatly reduce the work to be done.

2.1. *Von Mises yield criterion*

It will be convenient to write the von Mises yield criterion for the strain tensor $\overline{\overline{\epsilon}}$. Recall that the criterion sets the limit of elasticity (which we consider to be roughly the limit for breaking of the material) to

$$\operatorname{Tr} \left\{ \overline{\overline{\tau}}^2 \right\} \equiv J_2^* \geq \frac{2}{3} \sigma_{\text{yield}}^2, \tag{7}$$

[1] It is possible to derive a general form including the possibility of body forces, one more additional term would appear in the theorem.

[2] Thus we neglect the force of gravity inside of the rock.

with a material-specific constant σ_{yield}. For an isotropic material in the elastic regime, the deviatoric stress tensor $\overline{\overline{\tau}}$ is given by

$$\overline{\overline{\tau}} = \overline{\overline{\sigma}} - \frac{1}{3}\,\mathrm{Tr}\left\{\overline{\overline{\sigma}}\right\}\overline{\overline{1}} = 2\mu\overline{\overline{\epsilon}} - \frac{2\mu}{3}\,\mathrm{Tr}\left\{\overline{\overline{\epsilon}}\right\}\overline{\overline{1}}. \tag{8}$$

Show that the invariant J_2^* may be computed as

$$J_2^* = 4\mu^2\left[\mathrm{Tr}\left\{\overline{\overline{\epsilon}}^2\right\} - \frac{1}{3}\,\mathrm{Tr}\left\{\overline{\overline{\epsilon}}\right\}^2\right]. \tag{9}$$

2.2. *Green's function for the problem*

The solution by Boussinesq

$$\overline{u}_{\text{Boussinesq}}(\overline{r}) \quad = \quad u_r(r,z)\overline{e}_r + u_z(r,z)\overline{e}_z \tag{10}$$

with components

$$u_r(r,z) \quad = \quad \frac{1}{4\pi\mu(\lambda+\mu)}\frac{\mu r^2 R - (\lambda+2\mu)r^2 z + \mu z^2(R-z)}{rR^3}, \tag{11}$$

$$u_z(r,z) \quad = \quad -\frac{1}{4\pi\mu(\lambda+\mu)}\frac{(2\lambda+3\mu)z^2 + (\lambda+2\mu)r^2}{R^3}, \qquad R \equiv \sqrt{r^2+z^2}, \tag{12}$$

is the Green's function for the problem, as it is the solution of a unit point load at the surface. As the equilibrium equations are linear, the full solution of our problem may be written as the convolution of the load function and the Green's function,

$$\overline{u}(\overline{r}) = p_0 \int_{x^2+y^2\leq r_0^2} \mathrm{d}x\,\mathrm{d}y\,\overline{u}_{\text{Boussinesq}}(\overline{r} - x\overline{e}_x - y\overline{e}_y). \tag{13}$$

2.3. *The form of the strain tensor*

The strain tensor $\overline{\overline{\epsilon}}$ for the Boussinesq solution has the form

$$\overline{\overline{\epsilon}}_{\text{Boussinesq}} = \begin{pmatrix} \epsilon_{rr} & 0 & \epsilon_{rz} \\ 0 & \epsilon_{\theta\theta} & 0 \\ \epsilon_{rz} & 0 & \epsilon_{zz} \end{pmatrix}, \tag{14}$$

with

$$\epsilon_{rr} = \partial_r u_r, \qquad \epsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \epsilon_{zz} = \partial_z u_z, \qquad \epsilon_{rz} = \frac{1}{2}(\partial_z u_r + \partial_r u_z). \tag{15}$$

To evaluate the convolution it is convenient to change the basis[3] from cylindrical to cartesian, using

$$\overline{e}_r = \cos\theta\overline{e}_x + \sin\theta\overline{e}_y, \qquad \overline{e}_\theta = -\sin\theta\overline{e}_x + \cos\theta\overline{e}_y. \tag{16}$$

Show that the total strain tensor at the symmetry axis $r=0$ is of the form

$$\overline{\overline{\epsilon}}(r=0,z) = \pi p_0 \int_0^{r_0} r'\,\mathrm{d}r'\,\left[\left(\epsilon_{rr}(r',z) + \epsilon_{\theta\theta}(r',z)\right)\left(\overline{e}_x\otimes\overline{e}_x + \overline{e}_y\otimes\overline{e}_y\right) + 2\epsilon_{zz}(r',z)\overline{e}_z\otimes\overline{e}_z\right]. \tag{17}$$

[3] A tensor $\overline{\overline{\epsilon}}$ of 2^{nd} rank in a 3-dimensional vector space is written using some set of coordinates as (using the summation convention) $\overline{\overline{\epsilon}} = \epsilon_{ij}\overline{e}_i \otimes \overline{e}_j$.

2.4. *Obtaining the invariant* J_2^*

Now you are ready to get the the J_2^* via Eq.(9). In order to even futher reduce the work, you may realize that the special form of $\overline{\overline{\epsilon}}$ which we recovered in Eq.(17) enables to compute only

$$\epsilon_{zz}^{\text{total}} \equiv \pi p_0 \int_0^{r_0} r' \, \mathrm{d}r' \, \epsilon_{zz}(r', z) \qquad \text{and} \qquad \mathrm{Tr}\left\{\overline{\overline{\epsilon}}^{\text{total}}\right\} \equiv \pi p_0 \int_0^{r_0} r' \, \mathrm{d}r' \; \mathrm{Tr}\left\{\overline{\overline{\epsilon}}(r', z)\right\} \tag{18}$$

and to get J_2^* as a function of those. Find the maximum of J_2^* as a function of depth z.

2.5. *Practical consequences*

Let us assume that under the building there is sandstone. Typical values for sandstone are $\lambda \approx 2\,\text{GPa}$, $\mu \approx 24\,\text{GPa}$, $\sigma_{\text{yield}} \approx 10\,\text{MPa}$. If we estimate that a single floor of a building would weight roughly the same as a layer of concrete (density $2400\,\text{kg.m}^{-3}$) of thickness $0.75\,\text{m}$, we get that a building with n floors would cause pressure $p_0 \approx n \times 18\,\text{kPa}$. Is there a practical restriction on height of the building arising from the von Mises criterion?

Exercise 3. *Two-dimensional elastic problem*

Your task is to find the stress potential (function of Airy) ϕ for the 2-dimensional elastic problem described by the boundary conditions

$$\begin{aligned}
\overline{T}_{\text{right}} &= \overline{\overline{\sigma}} \cdot \overline{e}_x &&= (\sigma_{xx}, \sigma_{xy}), &(19)\\
\overline{T}_{\text{left}} &= \overline{\overline{\sigma}} \cdot (-\overline{e}_x) &&= -(\sigma_{xx}, \sigma_{xy}), &(20)\\
\overline{T}_{\text{top}} &= \overline{\overline{\sigma}} \cdot \overline{e}_y &&= (\sigma_{xy}, \sigma_{yy}), &(21)\\
\overline{T}_{\text{bottom}} &= \overline{\overline{\sigma}} \cdot (-\overline{e}_y) &&= -(\sigma_{xy}, \sigma_{yy}). &(22)\\
& && &(23)
\end{aligned}$$

No body force is assumed in the task.