Exercise 1. *Tidal Tensor and Tidal Work*

In this exercise we want to study the tidal force acting on a planet B which is moving on an orbit around a very large planet (or sun) A. We work in the cartesian coordinates with the planet B in its origin and the planet A at $z = -D$, the z direction pointing away from the planet A.

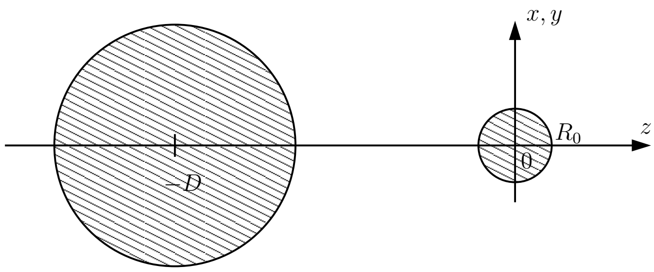

Figure 1: Geometry of the system

a) Suppose we have a particle with mass m at $\vec{r} = (x, y, z)$, find an expression for the graviational force $\vec{F}_g$ induced by planet B (the graviation of planet A is neglected)!

b) Typical values of $|\vec{r}|$ are much smaller than D. Therefore it is convenient to do a Taylor expansion of $\vec{F}_g$ about $\vec{r} = 0$ up to first order, i.e.

$$F_{g,i} = F_{g,i}(\vec{r} = 0) + \frac{\partial F_{g,i}}{\partial r_j}(\vec{r} = 0) \cdot r_j = F_{0,i} + T_{ij} r_j \quad (r_i = x, y, z) \tag{1}$$

where we introduced the tidal tensor T. Calculate the tidal tensor! Since the frame of reference is rotating the constant force $\vec{F}_0$ is compensated. Why? Show that the total force acting on the particle is given by $\vec{F}_{tot} = T\vec{r}$!

So far we considered particles with a mass m, next we want to go to the continuum where each volume element d^3r has a mass $dm = \rho d^3 r$ ($\rho \equiv const$). The force acting on them is $d\vec{F}_{tot}$ (where the former mass m has been replaced by $\rho d^3 r$).

c) The force $d\vec{F}_{tot}$ induces a strain on each particle of the planet A which gives rise to a deformation ($\vec{u} \neq 0$). The dynamics are described by the dynamical equilibrium equation

$$\rho d^3 r \frac{d\vec{u}}{dt} = d\vec{F}_{tot}. \tag{2}$$

Show that the volume change $dV/V = \nabla \cdot \vec{u}$ is constant for all t! What do you conclude if $dV/V = 0$ at $t = t_0$?

Hint: Take the divergence on both sides of the dynamical equilibrium equation.

As the planet deforms its shape, the infinitesimal volume elements move in the presence of a force, i.e. internal work is performed. Our goal is to compute this internal work. In order to do so, we introduce the infinitesimal tidal potential $d\phi$

$$d\phi = -\frac{1}{2}\vec{r}^T T' \vec{r} \rho d^3 r \quad (T' = T/m \text{ from (b)}) \tag{3}$$

of the volume element d^3r. The internal work is then given by the difference of the total tidal potential between the undisturbed and the disturbed planet shape. Thus, we need a parametrization of the volume which allows us to describe distortions but such that the volume is constant.

d) Verify that the tidal potential induces the total force, i.e. $-\nabla d\phi = d\vec{F}_{tot}$!

e) Let us consider the following parametrization of the planet in spherical coordinates

$$r_\gamma(\theta, \phi) = R_0 + R_0\gamma\left(3\cos^2\theta - 1\right). \tag{4}$$

Show that the volume V_γ enclosed by r_γ is independent of γ up to first order in γ,

$$V_\gamma = \int d\Omega \int_0^{r_\gamma(\theta,\phi)} dr\ r^2 = V_{\gamma=0} + \mathcal{O}(\gamma^2) \tag{5}$$

Hint: In order to avoid to compute all the following angular integrals, use the so-called spherical harmonic functions $Y_{lm}(\theta, \phi)$! Two examples of spherical harmonic functions are given by

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \quad , \quad Y_{20} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right). \tag{6}$$

These functions have a very useful property, namely they fulfill

$$\int d\Omega\ Y_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'}. \tag{7}$$

Thus, express all functions on θ, ϕ in terms of spherical harmonics and use their property in order to calculate the angular integrals!

f) The total potential ϕ_γ corresponding to the shape r_γ can be written as

$$\phi_\gamma = \int_{V_\gamma} d\phi. \tag{8}$$

Verify and find β such that

$$d\phi = \beta r^2 Y_{20}(\theta, \phi)\underbrace{d\phi d\theta\ \sin\theta\ dr\ r^2}_{=d\Omega dr r^2} \tag{9}$$

and compute

$$\phi_\gamma = \int d\Omega \int_0^{r_\gamma(\theta,\phi)} r^2 dr\ \beta r^2 Y_{20}(\theta, \phi)\ ! \tag{10}$$

g) Find an expression for $\Delta\phi_\gamma = \phi_{\gamma=0} - \phi_\gamma$ and replace the parameter γ by the measurable quantity $\Delta h = r_{max} - r_{min}$! On earth (earth in orbit around the sun) we have $\Delta h \approx 18m$ while on Io (moon of Jupiter in orbit around Jupiter) we have $\Delta h = 100m$. Find the necessary data and calculate $\Delta\phi_h$ for the earth and Io! What happens with the energy which is released by the distortion of the planets shape?