The XXZ Spin chain

The Hamiltonian

We will study a spin chain consisting of L spin-$\frac{1}{2}$ particles. Let us introduce the usual raising and lowering spin operators $S^{\pm} = S^x \pm iS^y$, such that

$$\begin{aligned} &S^+|\uparrow\rangle = 0, && S^-|\uparrow\rangle = |\downarrow\rangle, && S^z|\uparrow\rangle = \tfrac{1}{2}|\uparrow\rangle \\ &S^+|\downarrow\rangle = |\uparrow\rangle, && S^-|\downarrow\rangle = 0, && S^z|\downarrow\rangle = -\tfrac{1}{2}|\downarrow\rangle. \end{aligned} \tag{1}$$

Then we consider the Hamiltonian

$$\mathcal{H} = \Delta\frac{JL}{4} - J\sum_i \tfrac{1}{2}(S_i^+S_{i+1}^- + S_i^-S_{i+1}^+) + \Delta S_i^zS_{i+1}^z, \tag{2}$$

where Δ is some real number and $S^a_{L+1} = S^a_1$. This spin chain is called the XXZ-spin chain and is integrable. We will solve its spectrum by using both the algebraic and coordinate Bethe Ansatz. Let us have a look at a symmetry of the Hamiltonian

- Show that $[\mathcal{H}, S^z] = 0$.

As a consequence, the number of spins up and down are preserved and we can restrict to a fixed number of them.

The coordinate Bethe Ansatz

We start by defining the reference state $|0\rangle$ which has all spins up

$$|0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle. \tag{3}$$

- Show that $|0\rangle$ is an eigenstate of the Hamiltonian with eigenvalue 0.

We now proceed with finding the eigenstate of the Hamiltonian in case one of the spins is flipped. Consider the state

$$\begin{aligned} |p\rangle &= \sum_n e^{ipn}S_n^-|0\rangle \\ &= e^{ip}|\downarrow\uparrow\uparrow\ldots\rangle + e^{2ip}|\uparrow\downarrow\uparrow\ldots\rangle + e^{3ip}|\uparrow\uparrow\downarrow\ldots\rangle + \ldots. \end{aligned} \tag{4}$$

- Prove that $|0\rangle$ is an eigenstate of the Hamiltonian with eigenvalue

$$E(p) = \tfrac{1}{2}J(2\Delta - e^{ip} + e^{-ip}) \tag{5}$$

  provided that $e^{ipL} = 1$, which is a consequence of the periodic boundary conditions.

So far, the discussion is very similar to that of the XXX spin chain. The dispersion relation is modified however. The Bethe Ansatz for the XXX spin chain also needed the scattering phase as a crucial ingredient. To determine this, we consider the two magnon state

$$|p_1, p_2\rangle = \sum_{n_1<n_2}\left[e^{i(p_1n_1+p_2n_2)} + Ae^{i(p_2n_1+p_1n_2)}\right]S_{n_1}^-S_{n_2}^-|0\rangle. \tag{6}$$

- By considering terms in the Ansatz for $|p_1, p_2\rangle$ argue that its eigenvalue simply is given by $E(p_1) + E(p_2)$.

- By considering the term $|\ldots\downarrow_n\downarrow_{n+1}\ldots\rangle$ in the Ansatz for the wave function, show that the scattering term is given by

$$A = -\frac{e^{i(p_1+p_2)} + 1 - 2\Delta e^{ip_2}}{e^{i(p_1+p_2)} + 1 - 2\Delta e^{ip_1}}. \tag{7}$$

The periodic boundary conditions imply the same Bethe equations as for the XXX spin chain:

$$e^{ip_k L} = \prod_{k\neq l} A(p_l, p_k). \tag{8}$$

We can simplify the Bethe equations by making the following coordinate transformation

$$\Delta = \cos\hbar, \qquad e^{ip} = \frac{\sinh\hbar(u+\frac{i}{2})}{\sinh\hbar(u-\frac{i}{2})}. \tag{9}$$

- Check that the Bethe equations in the new coordinates are of the form

$$\left[\frac{\sinh\hbar(u_k+\frac{i}{2})}{\sinh\hbar(u_k-\frac{i}{2})}\right]^L = \prod_{l\neq k}\frac{\sinh\hbar(u_k-u_l+i)}{\sinh\hbar(u_k-u_l+i)}. \tag{10}$$

In this parameterization we immediately recover the XXX Bethe equations by sending $\hbar\to 0$.

The algebraic Bethe Ansatz

To make the algebraic Bethe Ansatz work we need the R-matrix and Lax operator. The R-matrix we consider is the following

$$R_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sinh\hbar(u_1-u_2)}{\sinh\hbar(u_1-u_2+i)} & \frac{\sinh\hbar i}{\sinh\hbar(u_1-u_2-i)} & 0 \\ 0 & \frac{\sinh\hbar i}{\sinh\hbar(u_1-u_2+i)} & \frac{\sinh\hbar(u_1-u_2)}{\sinh\hbar(u_1-u_2-i)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

- Check that R satisfies the Yang-Baxter equation.

The Lax operator is

$$L_{j,a}(u) = \begin{pmatrix} \frac{1+\tau}{2} + \frac{1-\tau}{2}\sigma_j^z & \rho\sigma_j^- \\ \rho\sigma_j^+ & \frac{1+\tau}{2} - \frac{1-\tau}{2}\sigma_j^z \end{pmatrix}, \tag{12}$$

where

$$\tau = \frac{\sinh\hbar(u-\frac{i}{2})}{\sinh\hbar(u+\frac{i}{2})}, \qquad \rho = \frac{\sinh\hbar i}{\sinh\hbar(u+\frac{i}{2})}, \tag{13}$$

Notice that as a matrix $L_{j,a}(u) = R(u-\frac{i}{2})$.

- Argue that L satisfies the fundamental commutation relations $R_{ab}L_{n,a}L_{n,b} = L_{n,b}L_{n,a}R_{ab}$.

We define the monodromy matrix as

$$T_a = L_{L,a}(u-u_L)\ldots L_{1,a}(u-u_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{14}$$

The corresponding transfer matrix is simply $t = A + D$. The next step is to define a ground state. We use the same ground state $|0\rangle$ as in the coordinate Bethe Ansatz

- Show that

$$A|0\rangle = |0\rangle, \qquad C|0\rangle = 0, \qquad D|0\rangle = \tau^L|0\rangle, \tag{15}$$

We will use the operator B to create magnons. In order to calculate the action of the transfer matrix on an M-magnon state we need to study the commutation relations.

- Work out the FCR to get

$$B(u)B(v) = B(v)B(u) \tag{16}$$

$$A(u)B(v) = \frac{\sinh \hbar(u-v-i)}{\sinh \hbar(u-v)} B(v)A(u) + \frac{\sinh \hbar i}{\sinh \hbar(u-v)} B(u)A(v) \tag{17}$$

$$D(u)B(v) = \frac{\sinh \hbar(u-v+i)}{\sinh \hbar(u-v)} B(v)D(u) - \frac{\sinh \hbar i}{\sinh \hbar(u-v)} B(u)D(v). \tag{18}$$

We define the M-magnon state to be

$$|u_1 \dots u_M\rangle = B(u_1)\dots B(u_M)|0\rangle. \tag{19}$$

- Show that $|u_1 \dots u_M\rangle$ is an eigenstate of the transfer matrix with eigenvalue

$$\Lambda = \prod_{m=1}^{M} \frac{\sinh \hbar(u-u_m-i)}{\sinh \hbar(u-u_m)} + \frac{\sinh \hbar(u-\frac{i}{2})}{\sinh \hbar(u+\frac{i}{2})}^{L} \prod_{m=1}^{M} \frac{\sinh \hbar(u-u_m+i)}{\sinh \hbar(u-u_m)} \tag{20}$$

  provided that the rapidities satisfy the Bethe equations

$$\left[\frac{\sinh \hbar(u_k+\frac{i}{2})}{\sinh \hbar(u_k-\frac{i}{2})}\right]^{L} = \prod_{l\neq k} \frac{\sinh \hbar(u_k-u_l+i)}{\sinh \hbar(u_k-u_l+i)}. \tag{21}$$

One can once again show that the Hamiltonian is given by the logarithmic derivative of the transfer matrix.