The Hénon-Heiles Potential

In 1964 Hénon and Heiles, inspired by motion of stars in galaxies, introduced the following Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) - q_2(q_1^2 + \tfrac{e}{3}q_2^2). \tag{1}$$

It turns out that this describes an integrable model only for $e = 1, 6, 16$.

Conserved quantities

We start by studying the additional conserved quantities.

- Show that for $e = 1$ the Hamiltonian splits into two identical uncoupled Hamiltonians. Thus, for $e = 1$, the model is trivially integrable as the energy for both subsystems is conserved.

We will now derive the second integral of motion for the other two cases and motivate why these cases are special. We start from the equations of motion and rewrite them

- Derive the equations of motion

$$\ddot{q}_1 = 2q_1q_2, \qquad\qquad \ddot{q}_2 = q_1^2 + eq_2^2. \tag{2}$$

- Solve the first equation for q_2 and introduce the new variable $\psi = (2q_1)^{-1}\dot{q}_1$. Show that the ψ satisfies the following ordinary differential equation

$$\frac{d^4\psi}{dt^4} + 2(6-e)\frac{d\psi}{dt}\frac{d^2\psi}{dt^2} - 4(e+4)\left[\psi^2\frac{d^2\psi}{dt^2} + \psi\left(\frac{d\psi}{dt}\right)^2\right] + 16e\psi^5 = 0. \tag{3}$$

Let us now have a look at transformations that leave (3) invariant. In particular, we consider a rescaling $\psi \to \psi' = \lambda\psi$ and also change $e \to e'$.

- From the definition of ψ show that the transformations reduce to $q_1 \to q_1' = Aq_1^\lambda$ where A is constant.
- Show that (3) is invariant under rescaling provided that

$$\lambda(6-e') = 6-e, \qquad \lambda^2(e'+4) = e+4, \qquad \lambda^4 e' = e. \tag{4}$$

This set of equations has four different solutions. There is the case for which $\lambda = 1$ corresponding to a trivial transformation. However, the other solutions correspond to the integrable cases:

$$(\lambda, e, e') = (-\tfrac{1}{2}, 1, 16), \qquad (\lambda, e, e') = (-1, 6, 6), \qquad (\lambda, e, e') = (-2, 16, 1). \tag{5}$$

We see that the cases $e = 1$ and $e = 16$ are related to each other by a rescaling of ψ. Futhermore, when $e = 6$ the equation is self-dual since it is invariant under $\psi \to -\psi$.

In fact, the constant A plays the role of this second conserved quantity. Indeed, consider the case $e = 6$. From the second equation of motion we find

$$\dddot{\psi} + 4(\dot{\psi}^2 + \psi\ddot{\psi}) = q_1^2 + 6(\dot{\psi} + 2\psi^2)^2 \tag{6}$$

Adding the tranformed equation to this eliminates the $\dddot{\psi}$ term and gives

$$q_1^2 + \frac{A^2}{q_1^2} = 8\psi\ddot{\psi} - 48\psi^4 - 4\dot{\psi}^2. \tag{7}$$

The above equation can be solved for A. Since A is constant it provides a non-trivial constant of motion.

- By expressing ψ and its derivatives in terms of q_i and p_i, show that for $e = 6$ the second conserved quantity is given by

$$F = 4p_1(q_1p_2 - q_2p_1) - (4q_2^2 + q_1^2)q_1^2, \tag{8}$$

and check that it Poisson commutes with the Hamiltonian.

- Now, by similar arguments show that

$$F = 3p_1^4 + 4q_1^3p_1q_1 - 12q_2q_1^2p_1^2 - 4q_1^4q_2^2 - \tfrac{2}{3}q_1^6, \tag{9}$$

is the conserved quantity when $e = 16$.

Laix pair

In case $e = 6$ it is possible to find a Lax pair in which the matrices are 2×2 and depend on a spectral parameter λ.

- Show that

$$L = \begin{pmatrix} 2\lambda p_2 + p_1q_1 & 4\lambda^2 - 4\lambda q_2 - q_1^2 \\ 4\lambda^3 + 4\lambda^2 q_2 + \lambda(q_1^2 + 4q_2^2) + p_1^2 & -2\lambda p_2 - q_1p_1 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ \lambda + 2q_2 & 0 \end{pmatrix} \tag{10}$$

forms a Lax pair for the Hénon-Heiles Hamiltonian when $e = 6$.

- Rederive the Hamiltonian and the conserved quanty (8) from the Lax pair.

In classical mechanics, a Lax pair that depends on a spectral parameter is equivalent to a Lax pair without a spectral parameter. The spectral parameter allows for a more compact presentation.

- Show that for $e = 6$ we have also have the Lax pair

$$L = \begin{pmatrix} p_1q_1 & -q_1^2 & 2p_2 & -4q_2 & 0 & 4 & 0 & 0 \\ p_1^2 & -q_1p_1 & q_1^2+4q_2^2 & -2p_2 & 4q_2 & 0 & 4 & 0 \\ 0 & 0 & p_1q_2 & -q_1^2 & 2p_2 & -4q_2 & 0 & 4 \\ 0 & 0 & p_1^2 & -q_1p_1 & q_1^2+4q_2^2 & -2p_2 & 4q_2 & 0 \\ 0 & 0 & 0 & 0 & p_1q_2 & -q_1^2 & 2p_2 & -4q_2 \\ 0 & 0 & 0 & 0 & p_1^2 & -q_1p_1 & q_1^2+4q_2^2 & -2p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_1q_2 & -q_1^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_1^2 & -q_1p_1 \end{pmatrix} \tag{11}$$

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2q_2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2q_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2q_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2q_2 & 0 \end{pmatrix} \tag{12}$$

You can identify the blocks as the coefficients for the different powers of λ.

- Show that the 8×8 Lax pair also gives rise to the first integrals.