Introduction to integrability

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Contents

1 Integrable classical Hamiltonian systems

1.1 Defining integrability

A Hamiltonian 1d system is comprised of

- A set of fields $u^a(x,t)$, $a = 1,2,\dots,m$ depending on a 1 dimensional space coordinate x and a time t. The phase space $\mathcal{M}$ is the set of points specifying the configuration of these fields at every given time.
- A Poisson bracket between phase space functionals

$$\{F,G\} = \int dxdy\, \omega^{ab}(x,y|u) \frac{\delta F[u]}{\delta u^a(x)} \frac{\delta G[u]}{\delta u^b(y)} \ ,$$

  i.e. functionals of the type $F[u] = \int dx\, f(u,u_x,u_{xx},\dots)$. The "matrix elements" $\omega^{ab}(x,y|u)$ of the Poisson bracket are typically of the form

$$\omega^{ab}(x,y|u) = \sum_n P_n(u,u_y,u_{yy},\dots)\partial_x^n \delta(x-y)$$

  for local Hamiltonian systems. The Poisson-bracket must satisfy the usual conditions: be antisymmetric, satisfy the Leibniz rule and the Jacobi identity.

  Example. *Frequently encountered Poisson brackets*

$$\{u(x),u(y)\}_1 = \partial_x \delta(x-y) \tag{1.1}$$

$$\{u(x),u(y)\}_2 = [2(u(x)+u(y))\partial_x + \partial_x^3]\delta(x-y) \tag{1.2}$$

$$\{u^a(x),u^b(y)\} = [f^{ab}{}_c u^c(y) + \eta^{ab}\partial_x]\delta(x-y) \ , \tag{1.3}$$

where η^{ab} is a symmetric matrix and $f^{ab}{}_c$ are antisymmetric in a, b and satisfy the Jacobi identities.

- A Hamitonian $H[u] = \int dx\, h(u, u_x, u_{xx}, \ldots)$ defining the time evolution

$$\dot{u}^a(x) = \{u^a(x), H[u]\} \ .$$

Example. *Hamiltonian systems which do not arise from a Lagrangian*

$$\begin{aligned} \text{mKdV}: \quad & \{\cdot,\cdot\}_1 \ , \ H[v] = \int dx\, \tfrac{1}{2}(v^4 - v_x^2) \quad \rightarrow \quad v_t = 6v^2 v_x + v_{xxx} \\ \text{KdV}: \quad & \{\cdot,\cdot\}_2 \ , \ H[u] = \int dx\, \tfrac{1}{2}u^2 \quad \rightarrow \quad u_t = 6uu_x + u_{xxx} \end{aligned}$$

Field theories formulated in terms of a Lagrangian have a canonical momentum, Poisson structure and Hamiltonian. If $\mathcal{L}$ is the Lagrangian density, then the momentum, Poisson bracket and Hamiltonian are defined by

$$\pi_a := \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a} \ , \quad \{\phi^a(x), \pi_b(y)\} := \delta^a_b \delta(x-y) \ . \quad H[\phi, \pi] := \int dx\, (\pi_a \dot{\phi}^a - \mathcal{L})$$

The e.o.m. take the usual form

$$\dot{\phi}^a(x) = \frac{\delta H[\phi,\pi]}{\delta \pi^a(x)} \ , \qquad \dot{\pi}^a(x) = -\frac{\delta H[\phi,\pi]}{\delta \phi^a(x)} \ .$$

Example. *The sine-Gordon field theory is defined by the Lagrangian*

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{\beta^2}(1 - \cos\beta\phi) = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{m^2}{2}\phi^2 + \tfrac{\beta^2 m^2}{4!}\phi^4 + \cdots \ .$$

The momentum and canonical Poisson structure are

$$\pi = \dot{\phi} \ , \qquad \{\phi(x), \dot{\phi}(y)\} = \delta(x-y) \ ,$$

while the Hamiltonian and e.o.m. are

$$\begin{aligned} H[\phi,\pi] &= \int dx\, \left[\frac{1}{2}(\pi^2 + \phi_x^2) + \frac{m^2}{\beta^2}(1-\cos\beta\phi)\right] \ , \\ \ddot{\phi} = \dot{\pi} &= \{\pi, H[\phi,\pi]\} = \phi_{xx} - \frac{m^2}{\beta}\sin\beta\phi \ . \end{aligned} \tag{1.4}$$

We shall also use the light cone coordinates

$$\begin{aligned} \tau &= \frac{t+x}{2} \ , \qquad \partial_\tau = \partial_t + \partial_x \ , \\ \sigma &= \frac{t-x}{2} \ , \qquad \partial_\sigma = \partial_t - \partial_x \ . \end{aligned} \tag{1.5}$$

in which the Lagrangian takes the form

$$\mathcal{L} = \phi_\tau \phi_\sigma - \frac{m^2}{\beta^2}(1 - \cos\beta\phi) \ ,$$

the Poisson structure is defined by

$$\pi = \frac{\partial \mathcal{L}}{\partial \phi_\tau} = \phi_\sigma \ , \qquad \{\phi(\sigma), \phi_\sigma(\sigma')\} = \delta(\sigma - \sigma')$$

and the Hamiltonian structure takes a particularly simple form

$$H[\phi] = \int dx \, \frac{m^2}{\beta^2}(1 - \cos \beta\phi) \ ,$$

$$\phi_{\tau\sigma} = \pi_\tau = \{\pi, H[\phi]\} = -\frac{m^2}{\beta} \sin \beta\phi \ .$$

Let us denote by $u(x) = \{u^a(x)\}$ the fields parametrizing the phase space of a 1d Hamiltonian system. A local integral of motion is a phase space functional $I[u]$ which is constant in time $\dot{I} = \{I, H\} = 0$ and can be presented in the form

$$I[u] = \int dx \, P(x|u, u_x, u_{xx}, \dots) \ ,$$

where $P(x)$ is a function of u and its derivatives evaluated at x. The density P will give rise to a local integral of motion if it is the time component of a conserved current

$$\frac{\partial}{\partial t} P(x|u, u_x, \dots) = \frac{\partial}{\partial x} Q(x|u, u_x, \dots) \ .$$

Thus, local integrals of motion are naturally associated to conserved currents.

There is no unanimously accepted definition of an integrable Hamiltonian system. A definition that works well in practical applications generalizes the concept of Liouville integrability in classical mechanics.

Definition. *We shall say that a (local) 1d Hamiltonian system is integrable if it has infinitely many local integrals of motion in involution.*

If one pushes the analogy with classical mechanics even further, then one must also require that the symplectic form vanishes on the level set specified by the i.o.m. The lack of preciseness is however irrelevant, because there is no analog to the Liouville theorem of classical mechanics , i.e. it is not possible to solve the e.o.m. in a trivial way once the i.o.m. are given. The above definition must be understood as a practical recipe for identifying integrable systems; usually, if one can find just a few i.o.m. besides the Hamiltonian, then it is almost certain that we are dealing with an integrable system. Let us see of few examples of i.o.m. in non-trivial Hamiltonian systems and get a feeling of how to search for them.

Example. *The mKdV and KdV equations are invariant w.r.t. the rescalings*

$$\text{mKdV}: \qquad t \mapsto a^3 t \ , \quad x \mapsto ax \ , \quad v \mapsto a^{-1} v \tag{1.6}$$

$$\text{KdV}: \qquad t \mapsto a^3 t \ , \quad x \mapsto ax \ , \quad u \mapsto a^{-2} u \tag{1.7}$$

We now introduce a degree corresponding to this rescaling

$$\text{mKdV}: \qquad \deg(v) = 1 \ , \quad \deg(\partial_x) = 1$$

$$\text{KdV}: \qquad \deg(u) = 2\ , \quad \deg(\partial_x) = 1$$

and classify the i.o.m. by this degree. For example, $P_n(x|v, v_x, v_{xx}, \dots)$ *is a density of degree* l *if it transform homogeneously under the rescaling* (1.7) *as*

$$P_l(x|v, v_x, v_{xx}, \dots) \mapsto a^{-l} P_l(x|v, v_x, v_{xx}, \dots)\ .$$

One can easily check using the mKdV e.o.m. that the following expressions

$$\begin{aligned}
P_1 &= v \\
Q_3 &= 2v^3 + v_{xx} \\
P_2 &= \tfrac{1}{2}v^2 \\
Q_4 &= \tfrac{3}{2}v^4 + v v_{xx} - \tfrac{1}{2}v_x^2 \\
P_4 &= \tfrac{1}{4}(v^4 - v_x^2) \\
Q_6 &= v^6 + v^3 v_{xx} - 3v^2 v_x^2 - \tfrac{1}{2}v_x v_{xxx} + \tfrac{1}{4}v_{xx}^2 \\
P_6 &= \tfrac{1}{6}v^6 - \tfrac{1}{6}v^2 v_x^2 + \tfrac{2}{9}v^3 v_{xx} + \tfrac{1}{12}v_{xx}^2
\end{aligned} \tag{1.8}$$

define conserved currents and charges by

$$\partial_t P_l = \partial_x Q_{l+2} \quad \to \quad I_l = \int dx\, P_{l+1}\ .$$

Similarly, the expressions bellow

$$\begin{aligned}
P_2 &= u \\
Q_4 &= 3u^2 + u_{xx} \\
P_4 &= \tfrac{1}{2}u^2 \\
Q_6 &= 2u^3 + u u_{xx} - \tfrac{1}{2}u_x^2 \\
P_6 &= \tfrac{1}{3}u^3 - \tfrac{1}{6}u_x^2 \\
Q_8 &= \tfrac{3}{2}u^4 + u^2 u_{xx} - 2u u_x^2 - \tfrac{1}{3}u_x u_{xxx} + \tfrac{1}{6}u_{xx}^2\ .
\end{aligned} \tag{1.9}$$

define conserved charges of the KdV system $I_l = \int dx\, P_{l+1}$.

The integrals of motion can be searched systematically by making the most general ansatz for P_l *and* Q_{l+2}*, which is compatible with the assigned degree. One can prove that (m)KdV has a unique non-trivial local integral of motion at every odd degree* l*. This set is complete and its elements are in involutions w.r.t. each other. We shall later use a Lax pair approach to generate these i.o.m. and especially to prove their involutivity.*

Example. *Consider the Klein-Gordon equation*

$$\phi_{\tau\sigma} = -m^2 V'(\phi)$$

for a general potential $V(\phi)$*. This equation is invariant w.r.t. the rescaling*

$$\tau \mapsto a\tau\ , \quad \sigma \mapsto a\sigma\ , \quad m \mapsto m a^{-1}\ , \quad \phi \mapsto \phi\ .$$

Introducing a grading

$$\deg \partial_\tau = \deg \partial_\sigma = \deg m = 1 \ , \quad \deg \phi = 0$$

we can organize the conserved currents by their degree, i.e.

$$\partial_\tau P_l(\sigma|\phi_\sigma, \phi_{\sigma\sigma}, \dots) = \partial_\sigma Q_l(\sigma|\phi_\sigma, \phi_{\sigma\sigma}, \dots) \ ,$$

where P_l *and* Q_l *are homogeneous polynomials of degree* l*. At degree 2 there is always a conserved current corresponding to the energy-momentum tensor*

$$\partial_\tau \left(\tfrac{1}{2}\phi_\sigma^2\right) = \partial_\sigma \left(-m^2 V(\phi)\right) \ .$$

Making the most general ansatz for P_3 *up to total* σ*-derivatives*

$$\partial_\tau(\tfrac{1}{3}\phi_\sigma^3) = \phi_\sigma^2\phi_{\sigma\tau} = -\phi_\sigma^2 V'(\phi)$$

we explicitly see that it is not possible to get a conserved current when $V \neq 0$*. At degree 4, the most general ansatz for* P_4*, up to total* σ*-derivatives, gives*

$$\begin{aligned}\partial_\tau(\tfrac{1}{4}\phi_\sigma^4 + \alpha\phi_{\sigma\sigma}^2) = (\phi_\sigma^3 + 2\alpha\phi_{\sigma\sigma}\partial_\sigma)\phi_{\sigma\tau} &= -m^2\left[\phi_\sigma^3 V' + 2\alpha\phi_{\sigma\sigma}\phi_\sigma V''\right] \\ &= \partial_\sigma\left(-m^2\phi_\sigma^2 V\right) + m^2(V - \alpha V'')\partial_\sigma\left(\phi_\sigma^2\right) \ .\end{aligned}$$

Thus, a non-trivial integral exists only for the potentials of the form

$$V(\phi) = a\, e^{\frac{\phi}{\sqrt{\alpha}}} + b\, e^{-\frac{\phi}{\sqrt{\alpha}}} + \text{const} \ .$$

By a rescaling and shift of ϕ*, this can be brought to the Sine-Gordon*

$$V_{SG} = \frac{(1-\cos\beta\phi)}{\beta^2} \ , \qquad V_{ShG} = \frac{\cosh b\phi}{b^2} \ , \qquad V_L = e^{2b\phi} \ .$$

Higher order conserved currents can be searched in a similar way. Here is what we get up to degree 6 for the sine-Gordon model

$$\begin{aligned} P_2 &= \tfrac{1}{2}\varphi_\sigma^2 \\ Q_2 &= m^2\cos\varphi \\ P_4 &= \tfrac{1}{4}\varphi_\sigma^4 - \varphi_{\sigma\sigma}^2 \\ Q_4 &= m^2\varphi_\sigma^2\cos\varphi \\ P_6 &= \tfrac{1}{6}\varphi_\sigma^6 - \tfrac{2}{3}\varphi_\sigma^2\varphi_{\sigma\sigma}^2 + \tfrac{8}{9}\varphi_\sigma^3\varphi_{\sigma\sigma\sigma} + \tfrac{4}{3}\varphi_{\sigma\sigma\sigma}^2 \\ Q_6 &= m^2\cos\varphi\left(\tfrac{1}{9}\varphi_\sigma^4 - \tfrac{4}{3}\varphi_{\sigma\sigma}^2\right) \ , \end{aligned} \tag{1.10}$$

where $\varphi = \beta\phi$*.*

An integrable hierarchy is comprised of

- A phase space $\mathcal{M}$ parametrized by some fields $u(x) = \{u^a(x)\}$ endowed with a Poisson-bracket $\{\cdot,\cdot\}$
- Infinitely many integrals of motion $\{I_n\}_{n\in\mathbb{N}}$ in involution $\{I_m, I_n\} = 0$

- A time evolution associated to every i.o.m.

$$\frac{\partial}{\partial t_m} u(x) = \{u(x), I_m[u]\} \ , \qquad m = 1, 2, 3, \cdots \ .$$

The different time evolutions are compatible with each other because

$$\frac{\partial}{\partial t_m}\frac{\partial}{\partial t_n} u(x) - \frac{\partial}{\partial t_n}\frac{\partial}{\partial t_m} u(x) = \{\{u(x), I_n\}, I_m\} - \{\{u(x), I_m\}, I_n\} = \{u(x), \{I_n, I_m\}\} = 0 \ .$$

Example. *The first few equations of the mKdV hierarchy are*

$$\begin{aligned}
\frac{\partial}{\partial t_0} v(x) &= \{v(x), I_0[v]\} = 0 \\
\frac{\partial}{\partial t_1} v(x) &= \{v(x), I_1[v]\} = v_x \\
\frac{\partial}{\partial t_3} v(x) &= -2\{v(x), I_3[v]\} = v_{xxx} + 6v^2 v_x \\
\frac{\partial}{\partial t_5} v(x) &= \{v(x), I_5[v]\} = \partial_x \left(v^5 + \tfrac{5}{3} v v_x^2 + \tfrac{5}{3} v^2 v_{xx} + \tfrac{1}{6} v_{xxxx} \right)
\end{aligned}$$

The mKdV, KdV and sine-Gordon hierarchies can be identified by setting

$$u = v^2 \pm i v_\sigma \ , \qquad v = \tfrac{1}{2} \varphi_\sigma \ . \tag{1.11}$$

The $v \mapsto u$ map is called the Miura transformation. It maps a solution of mKdV to KdV

$$u_t - (u_{xxx} + 6u_x u) = (2v \pm i\partial_x)[v_t - (v_{xxx} + 6v^2 v_x)] \ .$$

The map $\phi \mapsto v$ does not send sine-Gordon solutions to mKdV solutions. The identification is at the level of i.o.m.

$$\int dx \, P_l^{mKdV}(x) \propto \int dx \, P_l^{KdV}(x) \propto \int dx \, P_l^{SG}(x) \ .$$

Moreover, the Poisson bracket of ϕ induces the Poisson brackets of v and u (up to an overall proportionality constant).

1.2 Classical sine-Gordon model

In this section we shall discuss certain remarkable particle like solutions characteristic of non-linear wave equations. These are called solitons or solitary traveling waves: *a soliton is a wave which is localized in space and preserves its shape over time.*

Although we shall concentrate our discussion solely on the sine-Gordon model (1.4), let us mention that there are many other integrable Hamiltonian system whit solitonic solutions. Let us just mention the famous KdV soliton

$$u(x, t) = \frac{2\chi^2}{\cosh^2 \chi(x + 4\chi^2 t)} \ .$$

1.2.1 One soliton solutions

In order to find the sine-Gordon soliton solutions, one can first search for static solution $\phi(x,t) = \phi(x)$ and then perform a boost

$$x \mapsto \frac{x - vt}{\sqrt{1 - v^2}}$$

in order to get a propagating solution. Thus, we are looking for the solutions of

$$\phi'' = \frac{m^2}{\beta} \sin \beta\phi \ ,$$

subject to the b.c.

$$\phi_x(\pm\infty) = 0 \ , \tag{1.12}$$

which is a consequence of the localization property of the soliton. But this is the e.o.m. of the classical mechanical system

$$h = \tfrac{1}{2}\phi_x^2 + \underbrace{\tfrac{m^2}{\beta^2}(\cos \beta\phi - 1)}_{U(\phi)}$$

which describes the motion of a particle in a periodic potential $U(\phi)$. The boundary condition for $\phi(x)$ is equivalent to the requirement that the particle approaches one of the maximums of the potential $U(\phi)$ at "time" $x \mapsto \pm\infty$, i.e.

$$\beta\phi(\pm\infty) \in 2\pi\mathbb{Z} \ . \tag{1.13}$$

With the b.c. (1.12, 1.13) we can easily evaluate the total energy h of the solution at $x \to \pm\infty$

$$h = \left[\tfrac{1}{2}\phi_x^2 + \tfrac{m^2}{\beta^2}(\cos \beta\phi - 1)\right]\Big|_{x \to \pm\infty} = 0 \ .$$

Hence

$$x - x_0 = \pm \int \frac{d\phi}{\sqrt{h - 2U(\phi)}} = \pm\frac{1}{m} \int \frac{\beta d\phi}{2 \sin \frac{\beta\phi}{2}} = \pm\frac{1}{m} \log \tan \frac{\beta\phi}{4} \ .$$

Inverting the dependence of x on ϕ and boosting it we get the desired soliton and anti-soliton solutions

$$\phi_s(x,t) = \frac{4}{\beta} \tan^{-1} e^{\gamma m(x - vt - x_0)} \ , \quad \phi_{\bar{s}}(x,t) = \frac{4}{\beta} \tan^{-1} e^{-\gamma m(x - vt - x_0)} \ , \tag{1.14}$$

where $\gamma = 1/\sqrt{1 - v^2}$. In order to compute the dispersion relation of this solution we need the energy momentum tensor

$$T_\mu{}^\nu = \partial_\mu\phi \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi} - \delta_\mu^\nu \mathcal{L} \ .$$

The energy and momentum of a classical solution $\phi(x,t)$ are the defined by

$$H[\phi, \dot{\phi}] = \int dx^2 T_{00} = \int dx^2 \left[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi_x^2 + \frac{m^2}{\beta^2}(1 - \cos \beta\phi)\right] \ ,$$

$$P[\phi, \dot{\phi}] = \int dx^2 T_{01} = -\int dx^2 \dot{\phi}\phi_x \ .$$

Computing the dispersion relation of the soliton solution we get

$$H[\phi_s] = \frac{M}{\sqrt{1-v^2}} \ ,$$
$$P[\phi_s] = -\int dx\, \partial_t\phi_s \partial_x\phi_s = \frac{Mv}{\sqrt{1-v^2}} \ ,$$

which coincides with that of a relativistic particle of mass

$$M = M_s = M_{\bar{s}} = \frac{8m}{\beta^2} \ .$$

We recall that the dispersion relation of a relativistic massive particle takes a simpler form

$$E = M\cosh\theta \ , \qquad P = M\sinh\theta \ .$$

in terms of the rapidity θ defined by $v = \sinh\theta$. Notice that the antisoliton solution can be obtained from the soliton solution by the transformation

$$\theta \mapsto \theta + i\pi \ .$$

As we have seen, the solitons of the Sine-Gordon model interpolate between distinct minimums of the potential $V(\phi)$. In fact, solitonic solution exist also for non-integrable models like the ϕ^4 field theory

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - V(\phi) \ , \qquad V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2$$

The soliton and anti-soliton interpolating between the two minima $\phi_\pm = \pm m/\sqrt{\lambda}$ can be computed similarly

$$\phi_s(x,t) = \frac{m}{\sqrt{\lambda}}\tanh\frac{\gamma m(x-vt-x_0)}{\sqrt{2}} \ , \qquad \phi_{\bar{s}}(x,t) = -\frac{m}{\sqrt{\lambda}}\tanh\frac{\gamma m(x-vt-x_0)}{\sqrt{2}} \ .$$

Integrability manifest itself not in the existence of solitonic solutions, but in their interaction properties. To study these properties one needs multisolitonic solutions. By definition, these are classical solutions which at $t \to -\infty$ can be approximated by a superposition of well separated 1-soliton solutions. The question we are asking is: how do the multisolitonic solutions of integrable systems differ from those of non-integrable systems after the scattering has taken place, i.e. at $t \to \infty$?

1.2.2 Multisoliton solutions

In the sine-Gordon model the simplest way to generate multisolitonic solutions is via the Backlund transformation

$$\begin{aligned} \partial_\tau\phi_2 &= \partial_\tau\phi_1 + \frac{2m\eta}{\beta}\sin\frac{\beta}{2}(\phi_1+\phi_2) \\ \partial_\sigma\phi_2 &= -\partial_\sigma\phi_1 + \frac{2m}{\beta\eta}\sin\frac{\beta}{2}(\phi_1-\phi_2) \end{aligned} \qquad (1.15)$$

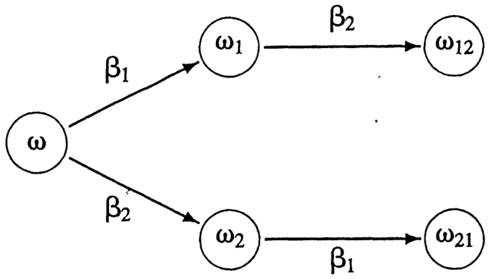

Figure 1: The two possibilities to perform the Backlund transformations of a solution ω with spectral parameters β_1 and β_2.

which maps a solution ϕ_1 to another solution ϕ_2. This can be sean by acting with ∂_σ on the first equation and then using the second one to evaluate it

$$\partial_\sigma \partial_\tau (\phi_2 - \phi_1) = m\eta \cos \frac{\beta}{2}(\phi_1 + \phi_2)\, \partial_\sigma (\phi_1 + \phi_2) = \frac{2m^2}{\beta} \cos \frac{\beta}{2}(\phi_1 + \phi_2) \times$$
$$\times \sin \frac{\beta}{2}(\phi_1 - \phi_2) = \frac{m^2}{\beta}(\sin \beta\phi_1 - \sin \beta\phi_2) \; .$$

Multisoliton solutions are generated by acting with the Backlung transformation interatively on the trivial solution $\phi = 0$. Thus, the 1-soliton solution can be recovered by setting $\phi_1 = 0$ in eq. (1.15) and integrating

$$\eta^{-1}\partial_\tau \phi_2 = -\eta \partial_\sigma \phi_2 = \frac{2m}{\beta} \sin \frac{\beta\phi_2}{2} \; .$$

The most general solution is

$$\phi_2 = \frac{4}{\beta} \tan^{-1} \exp m \left(\eta^{-1}\tau - \eta\sigma - x_0 \right) \; ,$$

where x_0 is a constant of integration. If we set $\eta = e^\theta$ and recall the definition of light cone coordinates (1.5) we recover the soliton solution (1.14)

$$\phi_s = \frac{4}{\beta} \tan^{-1} \exp \left(x \cosh\theta - t \sinh\theta - x_0 \right) \; . \tag{1.16}$$

We can generate 2-soliton solutions by inserting instead of ϕ_1 in eq. (1.15) the one soliton solution ϕ_s and then integrating the differential equation for ϕ_2. However, this last step is not completely trivial. In fact, there is a more elegant, *purely algebraic* way to generate multi-soliton solutions due to Bianchi.

Consider a classical solution ϕ of the sine-Gordon equation. We can generate two new solutions ϕ_1 and ϕ_2 by integrating the Backlund transformation of ϕ with spectral parameter η_1 and, respectively, η_2. If we now integrate the Backlund transformation of ϕ_1 with spectral parameter η_2 then we get a solution ϕ_{12}. Similarly, we denote by ϕ_{21} the Backlund transformation of ϕ_2 with spectral parameter η_1. We have represented these procedures by the diagram in fig. 1. In general, the solutions ϕ_{12} and ϕ_{21} will depend not only on η_1 and η_2, but also on some integration constants. Therefore, in general they $\phi_{12} \neq \phi_{21}$. However, we

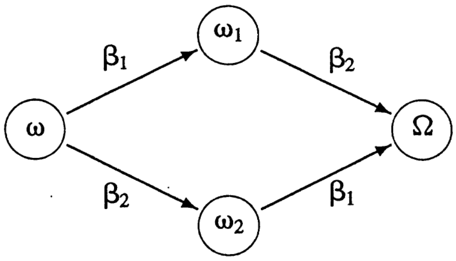

Figure 2: A commutative Bianchi diagram.

can ask the following question: is it possible to chose the integration constants in such a way that

$$\phi_{12} = \phi_{21} = \Phi?$$

The answer is yes! This is the Bianchi's permutability theorem, illustrated in fig. 2. To prove the theorem notice that if the solution Φ exists, then the commutativity of the Backlund transformations requires that

$$\begin{aligned} \partial_\tau \phi_{12} &= \partial_\tau \phi_1 + \frac{2m\eta_2}{\beta} \sin \frac{\beta}{2}(\Phi + \phi_1) = \partial_\tau \phi + \frac{2m\eta_1}{\beta} \sin \frac{\beta}{2}(\phi + \phi_1) + \\ &+ \frac{2m\eta_2}{\beta} \sin \frac{\beta}{2}(\Phi + \phi_1) \ , \\ \partial_\tau \phi_{21} &= \partial_\tau \phi_2 + \frac{2m\eta_1}{\beta} \sin \frac{\beta}{2}(\Phi + \phi_2) = \partial_\tau \phi + \frac{2m\eta_2}{\beta} \sin \frac{\beta}{2}(\phi + \phi_2) + \\ &+ \frac{2m\eta_1}{\beta} \sin \frac{\beta}{2}(\Phi + \phi_2) \ . \end{aligned} \tag{1.17}$$

Subtracting the two equations we get

$$\eta_1 \left[\sin \frac{\beta}{2}(\phi + \phi_1) - \sin \frac{\beta}{2}(\Phi + \phi_2) \right] = \eta_2 \left[\sin \frac{\beta}{2}(\phi + \phi_2) - \sin \frac{\beta}{2}(\Phi + \phi_1) \right] \ .$$

One of the solutions is given by

$$\Phi = \phi + \frac{4}{\beta} \tan^{-1} \left[\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} \tan \frac{\beta}{4}(\phi_1 - \phi_2) \right] \ . \tag{1.18}$$

To finish the proof of the theorem one simply checks that the above expression satisfies eqs. (1.17) together with

$$\begin{aligned} \partial_\sigma \phi_{12} &= -\partial_\sigma \phi_1 + \frac{2m}{\eta_2 \beta} \sin \frac{\beta}{2}(\phi_1 - \Phi) = \partial_\sigma \phi - \frac{2m}{\eta_1 \beta} \sin \frac{\beta}{2}(\phi - \phi_1) + \\ &+ \frac{2m\eta_2}{\beta} \sin \frac{\beta}{2}(\phi_1 - \Phi) \ , \\ \partial_\sigma \phi_{21} &= -\partial_\sigma \phi_2 + \frac{2m}{\eta_1 \beta} \sin \frac{\beta}{2}(\phi_2 - \Phi) = \partial_\sigma \phi - \frac{2m}{\eta_2 \beta} \sin \frac{\beta}{2}(\phi - \phi_2) + \\ &+ \frac{2m\eta_1}{\beta} \sin \frac{\beta}{2}(\phi_2 - \Phi) \ . \end{aligned}$$

Setting in eq. (1.18) $\phi = 0$, $\eta_1 = e^{\theta}$, $\eta_2 = e^{-\theta}$ and

$$\phi_i = \frac{4}{\beta} \tan^{-1} \exp m \left(\eta_i^{-1} \tau - \eta_i \sigma \right) \ ,$$

we get the soliton anti-soliton solution in the center of mass frame

$$\phi_{s\bar{s}}(x,t) = -\frac{4}{\beta} \tan^{-1} \frac{\sinh m\gamma v t}{v \cosh m\gamma x} \ . \tag{1.19}$$

On the other hand, for $\phi = 0$, $\eta_1 = e^{\theta}$, $\eta_2 = -e^{-\theta}$ we get the soliton-soliton solution in the center of mass frame

$$\phi_{ss}(x,t) = \frac{4}{\beta} \tan^{-1} \frac{v \sinh m\gamma x}{\cosh m\gamma v t} \ .$$

Let us consider the $s\bar{s}$ solution in more detail. At asymptotic times we have a superposition of a soliton and an anti-soliton

$$\phi_{s\bar{s}}(x,t) \equiv \begin{cases} \frac{4}{\beta} \tan^{-1} e^{-m\gamma[x+v(t-\Delta t)]} + \frac{4}{\beta} \tan^{-1} e^{m\gamma[x-v(t-\Delta t)]} \ , & t \to -\infty \\ \frac{4}{\beta} \tan^{-1} e^{-m\gamma[x+v(t+\Delta t)]} + \frac{4}{\beta} \tan^{-1} e^{m\gamma[x-v(t+\Delta t)]} \ , & t \to +\infty \end{cases} \ ,$$

where

$$\Delta t = -\frac{\log v}{m\gamma v}$$

is essentially (half) the phase shift of the soliton and anti-soliton waves after the scattering. The striking property of this solutions is that at *both* $t \to \pm\infty$ it can be approximated by the *same* superposition $s + \bar{s}$. This means that the scattering of solitons with antisolitons does not change their asymptotic shapes. In particular, notice that $s + \bar{s} \mapsto s + \bar{s}$ is the only scattering process of a soliton with an anti-soliton!

The energy and momentum of the $s\bar{s}$ solution in the center of mass frame is

$$E[\phi_{s\bar{s}}] = 2M \cosh\theta \ , \quad P[\phi_{s\bar{s}}] = 0 \ .$$

If we treat v in eq. (1.19) as a formal variable, then we can generate a new solution localized in the neighborhood of $x = 0$ by setting $v = i \tan\vartheta_b \in \mathbb{R}$

$$\phi_b(x,t) = -\frac{4}{\beta} \tan^{-1} \frac{\sinh(tm \sin\vartheta_b)}{\tan\vartheta_b \cosh(xm \cos\vartheta_b)} \ .$$

The above solution is called a breather. Its energy

$$E[\phi_b] = 2M \cos\vartheta_b \leq 2M$$

suggests that it is an $s\bar{s}$ bound state.

The general n soliton solution obtained by iterating n times the Backlund transformation with parameters $\eta_1 = e^{\theta_1}, \ldots, \eta_n = e^{\theta_n}$ has additive energy and momentum

$$E = \sum_i M \cosh\theta_i \ , \qquad P = \sum_i M \sinh\theta_i \ .$$

The higher spin charges are also additive. Let us consider the degree 3 charges

$$I_3^{\sigma} = \int dx \ \left[\tfrac{1}{4} \varphi_{\sigma}^4 - \varphi_{\sigma\sigma}^2 - m^2 \varphi_{\sigma}^2 \cos\varphi \right] \ ,$$

$$I_3^\tau = \int dx\ \left[\tfrac{1}{4}\varphi_\tau^4 - \varphi_{\tau\tau}^2 - m^2\varphi_\sigma^2 \cos\varphi\right]$$

arising from the conserved current (1.10). Evaluating them on the solitonic solution (1.16) we get

$$I_3^\sigma[\phi_s] = \frac{16m^3e^{3\theta}}{3}\ , \qquad I_3^\tau[\phi_s] = \frac{16m^3e^{-3\theta}}{3}\ .$$

In general, higher spin charges of multisolitonic solutions have the form

$$I_s^\pm \propto \sum_i e^{\pm s\theta_i}\ . \tag{1.20}$$

Thus, the conservation of all these charges during the scattering process prohibits soliton creation or annihilation. Moreover, the set of incoming rapidities must be a permutation of the set of outgoing rapidities.

Let us summarize the results we got for the classical sine-Gordon model. The sine-Gordon field ϕ does not correspond to a massive particle of the theory. The actual particles are the soliton and anti-soliton of mass $M = 8m/\beta^2$ together with the continuous spectrum of soliton-antisoliton bound states of mass $2M\cos\theta_b$. The presence of an infinite number of conservation laws (1.20) prohibits particle creation and requires that the set of outgoing rapidities is a permutation of the set of incoming rapidities. The only effect of the interaction is to change the internal phases of the waves carrying the solitonic particles. We shall see that many of these properties persist in the quantum theory.

1.3 Zero curvature representation

Our definition of a classical integrable system requires not only the existence of an infinite number of i.o.m., but also that they are in involution w.r.t. each other. The modern method to prove involutivity starts with the rewriting of the e.o.m. as the consistency condition

$$[\partial_t - U, \partial_x - V] = \partial_x U - \partial_t V + [U, V] = 0 \tag{1.21}$$

of an overdetermined system of linear equations

$$\partial_t\Psi(x,t|\lambda) = U(x,t|\lambda)\Psi(x,t|\lambda)\ , \qquad \partial_x\Psi(x,t|\lambda) = V(x,t|\lambda)\Psi(x,t|\lambda)\ . \tag{1.22}$$

The rewriting of e.o.m. in the form (1.21) is called the *zero curvature representation*, while the linear system (1.22) is called the *auxiliary problem*.

Suppose that x lives on a circle of radius 1. The *monodromy* of $\Psi(x,t)$ around the spacial circle

$$\Psi(2\pi, t|\lambda) = T(\lambda)\Psi(0,t|\lambda) \tag{1.23}$$

can be computed as a path ordered integral

$$T(\lambda) = \mathcal{P}\exp\int_0^{2\pi} dx\, V(x,t|\lambda)\ . \tag{1.24}$$

This is an important quantity that can be used to generate i.o.m. Indeed, the evolution of $T(t)$ is described by the following equation

$$\dot{T}(\lambda) = \int_0^{2\pi} dx\ \left(\mathcal{P}\exp\int_x^{2\pi} dy\, V\right)\dot{V}\left(\mathcal{P}\exp\int_0^{x} dy\, V\right)$$

$$= \int_0^{2\pi} dx \left(\mathcal{P} \exp \int_x^{2\pi} dy\, V \right) (\partial_x U + [U,V]) \left(\mathcal{P} \exp \int_0^x dy\, V \right)$$
$$= \int_0^{2\pi} dx\, \partial_x \left[\left(\mathcal{P} \exp \int_x^{2\pi} dy\, V \right) U \left(\mathcal{P} \exp \int_0^x dy\, V \right) \right] = [U(2\pi, t|\lambda), T(\lambda)] \ .$$

Therefore, the trace of the monodromy matrix

$$t(\lambda) = \mathrm{tr}\, T(\lambda) \ ,$$

called *transfer matrix*, is a conserved quantity. Upon expanding in λ it generates infinitely many i.o.m. We shall see in a moment around which point to perform the expansion in order to get *local* i.o.m.

Example. *The KdV equation can be represented as the zero curvature condition of the connexion*

$$U = \begin{pmatrix} -u_x & 4\lambda + 2u \\ 4\lambda^2 - 2\lambda u - u_{xx} - 2u^2 & u_x \end{pmatrix} \ , \quad V = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \ . \qquad (1.25)$$

Explicitly, eq. (1.21) *reads*

$$\partial_x U - \partial_t V + [U,V] = \begin{pmatrix} 0 & 0 \\ u_t - 6uu_x - u_{xxx} & 0 \end{pmatrix} \ .$$

Example. *The zero curvature representation for sine-Gordon is*

$$U = \begin{pmatrix} \frac{i\varphi_\tau}{4} & \frac{im}{2\lambda} e^{-\frac{i\varphi}{2}} \\ \frac{im}{2\lambda} e^{\frac{i\varphi}{2}} & -\frac{i\varphi_\tau}{4} \end{pmatrix} \ , \qquad V = \begin{pmatrix} -\frac{i\varphi_\sigma}{4} & \frac{im\lambda}{2} e^{\frac{i\varphi}{2}} \\ \frac{im\lambda}{2} e^{-\frac{i\varphi}{2}} & \frac{i\varphi_\sigma}{4} \end{pmatrix} \ , \qquad (1.26)$$

where we recall that $\varphi = \beta\phi$.

Now, the crucial ingredient to get i.o.m. in involution is to assume the existence of a "good" Poisson bracket. Suppose that the Poisson bracket of the dynamical fields of the model are such that the Poisson bracket of the V matrix elements can be written in the form

$$\{V_1(x|\lambda), V_2(y|\lambda)\} = \delta(x-y)[r_{12}(\lambda,\mu), V_1(x|\lambda) + V_2(x|\lambda)] \ , \qquad (1.27)$$

where $V_1 = V \otimes I$, $V_2 = I \otimes V$. This type of Poisson brackets are called *ultra-local.*[1] The matrix r_{12} which acts on the tensor product $\Psi \otimes \Psi$ is called the *classical r-matrix.* We assume it to be independent of x, y. Eq. (1.27) implies the following Poisson brackets for the monodromy matrix

$$\{T_1(\lambda), T_2(\mu)\} = [r_{12}(\lambda,\mu), T_1(\lambda)T_2(\mu)] \ , \qquad (1.28)$$

which is called the Sklyanin bracket. The proof is computational. We first write

$$\{T_{ij}(\lambda), T_{kl}(\mu)\} = \int_0^{2\pi} dx \int_0^{2\pi} dy \frac{\delta T_{ij}(\lambda)}{\delta V_{ab}(x|\lambda)} \frac{\delta T_{kl}(\mu)}{\delta V_{cd}(y|\mu)} \{V_{ab}(x|\lambda), V_{cd}(y|\mu)\} \ . \qquad (1.29)$$

[1] Ultra-local classical Hamiltonian system are important because there is a standard way to quantize them.

The variation of the monodromy is

$$\delta T(\lambda) = \int_0^{2\pi} dx \left(\mathcal{P} e^{\int_x^{2\pi} dz V(z|\lambda)} \right) \delta V(x|\lambda) \left(\mathcal{P} e^{\int_0^x dz V(z|\lambda)} \right) .$$

Introducing the transport matrix

$$T(x, y|\lambda) = \mathcal{P} \exp \int_x^y dz\, V(z|\lambda)$$

we can write

$$\frac{\delta T_{ij}(\lambda)}{\delta V_{ab}(x|\lambda)} = T_{ia}(2\pi, x|\lambda) T_{bj}(x, 0|\lambda) \ .$$

Hence, we can rewrite eq. (1.29) in tensor like notations as

$$\{T_1(\lambda), T_2(\mu)\} = \int_0^{2\pi} dxdy T_1(2\pi, x|\lambda) T_2(2\pi, y|\mu) \{V_1(x|\lambda), V_2(y|\mu)\} T_1(x, 0|\lambda) T_2(y, 0|\mu) \ .$$

Using the Poisson brackets (1.27) and eliminating V via the equations

$$\partial_x T(x, 0|\lambda) = V(x|\lambda) T(x, 0|\lambda) \ , \qquad \partial_x T(2\pi, x|\lambda) = -T(2\pi, x|\lambda) V(x|\lambda)$$

we arrive at the desired result

$$\begin{aligned} \{T_1(\lambda), T_2(\mu)\} &= \int_0^{2\pi} dx \partial_x \left[T_1(2\pi, x) T_2(2\pi, x) r_{12}(\lambda, \mu) T_1(x, 0) T_2(x, 0) \right] \\ &= \left[r_{12}(\lambda, \mu), T_1(\lambda) T_2(\mu) \right] \ . \end{aligned}$$

This implies the involution of transfer matrices

$$\{t(\lambda), t(\mu)\} = 0 \ .$$

Example. *Due to the presence of derivatives of the δ-function in the KdV Poisson bracket* (1.2)*, the matrix elements of $V(x|\lambda)$ cannot satisfy* (1.27)*. This problem does not occur in the sine-Gordon model. In this case, the classical r-matrix exists and is given by*

$$r(\lambda, \mu) = -\frac{1}{8(\lambda^2 - \mu^2)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda^2 + \mu^2 & -2\lambda\mu & 0 \\ 0 & -2\lambda\mu & \lambda^2 + \mu^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

Notice that $r(\lambda, \mu)$ depends only on the ratio λ/μ.

Notice that the existence of a classical r-matrix is not guaranteed *a priori*, because the Jacobi identity for the Sklyanin bracket requires a non-trivial consistency condition to hold

$$[r_{13}(\lambda, \nu), r_{23}(\mu, \nu)] + [r_{12}(\lambda, \mu), r_{13}(\lambda, \nu) + r_{23}(\mu, \nu)] = 0 \ .$$

This equation is called the classical Yang-Baxter equation.

1.4 Generating integrals of motion

In this section we would like to illustrate on the example of the sine-Gordon model how to use the zero curvature representation in order to generate infinitely many local i.o.m. This will prove the integrability of the model according to the definition given in sec. 1.1. The involutivity of i.o.m. is guaranteed by the existence of a classical r-matrix. We shall see that the sine-Gordon model has local as well as non-local i.o.m.

In order to simplify the calculation of the monodromy (1.24), let us perform a gauge transformation on V

$$g(\partial_\sigma - V)g^{-1} = \partial_\sigma - V^g \quad \Rightarrow \quad V^g = g_\sigma g^{-1} + gVg^{-1} \ .$$

The gauge transformed monodromy matrix can be computed as the unique solution of the equation

$$T^g_\sigma(\sigma, 0|\lambda) = V^g(\sigma|\lambda)T^g(\sigma, 0|\lambda)$$

with the initial condition $T^g(0, 0|\lambda) = I$. An explicit calculation shows that

$$T^g(\sigma, 0|\lambda) = g(\sigma)T(\sigma, 0|\lambda)g^{-1}(0)$$

is a solution. The periodicity of $g(\sigma)$ then implies that the transfer matrix is gauge invariant

$$t^g(\lambda) = t(\lambda) \ .$$

The gauge transformation

$$g = \begin{pmatrix} e^{\frac{i\varphi}{4}} & 0 \\ 0 & e^{-\frac{i\varphi}{4}} \end{pmatrix}$$

brings the matrix V for the sine-Gordon model (1.26) to a simpler form

$$V^g = \frac{\zeta}{2}\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix} \ , \qquad \zeta = im\lambda \ .$$

From the formula

$$T^g(\lambda) = \sum_{n=0}^{\infty} \int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \ldots \int_0^{\sigma_{n-1}} d\sigma_n V^g(\sigma_1|\lambda) \cdots V^g(\sigma_{n-1}|\lambda)V^g(\sigma_n|\lambda)$$

we get the desired Taylor expansion of the transfer matrix around $\lambda = 0$

$$t(\lambda) = \sum_{n=0}^{\infty} \left(\frac{\zeta}{2}\right)^{2n} J_{2n} \ ,$$

where J_{2n} are non-local i.o.m. explicitly given by

$$J_{2n} = \int_0^{2\pi} d\sigma_1 \ldots \int_0^{\sigma_{2n-1}} d\sigma_{2n} 2\cos\left[\sum_{k=1}^{2n} \varphi(\sigma_k)(-1)^k\right] \ .$$

In general, an expansion of the transfer matrix around a regular point of $V^g(\sigma|\lambda)$ will produce non-local i.o.m.

We shall drop the index g in the following. To generate local i.o.m. we should carry out a Laurent expansion around a pole of $V(\sigma|\lambda)$, in this case $\lambda = \infty$. We do this by first solving the auxiliary linear problem

$$\Psi_\sigma(\sigma|\lambda) = V(\sigma|\lambda)\Psi(\sigma|\lambda) \ . \tag{1.30}$$

Once the solution Ψ is known, we can compute $T(\lambda)$ with the help of eq. (1.23). In terms of the components $\Psi = (\psi, \tilde{\psi})^t$, eq. (1.30) reads

$$\psi_\sigma = \frac{\zeta}{2} e^{i\varphi} \tilde{\psi} \ , \quad \tilde{\psi}_\sigma = \frac{\zeta}{2} e^{-i\varphi} \psi \ .$$

or, equivalently,

$$\psi_{\sigma\sigma} - i\varphi_\sigma \psi_\sigma - \frac{\zeta^2}{4}\psi = 0 \ , \quad \tilde{\psi}_{\sigma\sigma} + i\varphi_\sigma \tilde{\psi}_\sigma - \frac{\zeta^2}{4}\tilde{\psi} = 0 \ . \tag{1.31}$$

Let us rewrite these equations in terms of the logarithmic derivatives $p = \psi_\sigma/\psi$ and $\tilde{p} = \tilde{\psi}_\sigma/\tilde{\psi}$ as

$$p_\sigma + p^2 - i\varphi_\sigma p - \frac{\zeta^2}{4} = 0 \ , \quad \tilde{p}_\sigma + \tilde{p}^2 + i\varphi_\sigma \tilde{p} - \frac{\zeta^2}{4} = 0 \ . \tag{1.32}$$

Notice that $\tilde{p}$ can be obtained from p by changing $\varphi_\sigma \mapsto -\varphi_\sigma$. From eq. (1.32) it follows that p has a Laurent expansion around $\zeta = \infty$ of the following form

$$p = \pm\frac{\zeta}{2} + \sum_{n=0}^{\infty} \frac{p_n}{\zeta^n} \ .$$

The coefficient functions p_n can be determined recursively by plugging this expansion in eq. (1.32). The two sign choices correspond to the two linearly independent solution of (1.31). Let us choose the + sign in the following. Then p is uniquely determined by the following recurrence relations

$$\begin{aligned} p_0 &= \frac{i\varphi_\sigma}{2} \\ p_1 &= p_0^2 - p_0' \\ p_{n+1} &= -p_n' - \sum_{k=1}^{n-1} p_k p_{n-k} \ , \end{aligned} \tag{1.33}$$

where $n \geq 1$ and primes denote σ-derivatives. Notice that all the p_n with $n \geq 1$ depend solely on p_1 and its derivatives. The first few terms are

$$\begin{aligned} p_2 &= -p_1' \\ p_3 &= -p_2' - p_1^2 = p_1'' - p_1^2 \\ p_4 &= -p_3' - 2p_1p_2 = (-p_1'' + 2p_1^2)' \\ p_5 &= -p_4' - p_2^2 - 2p_1p_3 = (p_1'' - 3p_1^2)'' + p_1'^2 - 2p_1^3 \quad \text{etc.} \end{aligned} \tag{1.34}$$

It is possible to prove that p_{2n} are total derivatives.

Returning to the original variables $\psi, \tilde{\psi}$, we can write

$$\psi(\sigma|\lambda) = \psi_1(0|\lambda) \exp \int_0^\sigma d\sigma p(\sigma|\lambda) \ , \quad \tilde{\psi}(\sigma|\lambda) = \tilde{\psi}_1(0|\lambda) \exp \int_0^\sigma d\sigma \tilde{p}(\sigma|\lambda) \ ,$$

which immediately gives the monodromy

$$T(\lambda) = \begin{pmatrix} e^{\int_0^{2\pi} d\sigma p(\sigma|\lambda)} & 0 \\ 0 & e^{\int_0^{2\pi} d\sigma \tilde{p}(\sigma|\lambda)} \end{pmatrix} . \tag{1.35}$$

Now, because the eigenvalues of the monodromy are constants of motion, we see that the arguments of the exponentials in eq. (1.35)

$$\int_0^{2\pi} d\sigma p(\sigma|\lambda) = \zeta\pi + \sum_{n=0}^{\infty} \frac{I_{2n+1}[\varphi_\sigma]}{\zeta^{2n+1}} ,$$

generate local intergrals of motion

$$I_{2n+1}[\varphi_\sigma] = \int_0^{2\pi} p_{2n+1}(\sigma) . \tag{1.36}$$

The integrals of motion generated from $\tilde{p}$ are not algebraically independent because of the relation $p\tilde{p} = \frac{\zeta^2}{4}$. Evaluating the first few local integrals (1.36) with the help of eqs. (1.34) we recover the conserved charges (1.10) (up to a proportionality coefficient).

Moreover, notice that the first two terms in eq. (1.33) establish the connection of sine-Gordon with mKdV via the identification $p_0 = iv$ and KdV via the identification $p_1 = -(v^2 + iv) = -u$ defined in eq. (1.11). Thus, we explicitly see that the conserved charges (1.36) agree with the mKdV charges (1.8) if we express the former in terms of v and with (1.9) if we express them in terms of u. In fact, this relation indirectly proves the involutivity of the mKdV and KdV i.o.m., which can not be proved directly with the techniques of sec. 1.3 due to non ultra-locality issues.