The Heterotic String

In the context of the proseminar *Conformal Field Theory and Strings*
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Abstract

The heterotic string has been proposed for the first time in [1]. It is a mixture (”hybris”) of the right-moving sector of the superstring and the left-moving sector of the bosonic string. The two sectors need different space-time dimensions to cancel the anomalies. The matching of dimensions is done by compactifying the exceeding ones on a compact manifold. In this case, a 16 dimensional torus is used. By considering modular invariance of the genus one string partition function, the possible shapes of the torus are restricted up to two. This translates in two possible gauge groups for the heterotic string theory: $E_8 \times E_8$ and $SO(32)$.

Contents

1 Introduction

This report is the base (and the deeper analysis of its contents) of the presentation carrying the same title, which was given in the context of the proseminar *Conformal Field Theory and Strings* during the spring semester 2013 at the ETHZ. The proseminar was coordinated by Prof. Dr. M. Gaberdiel.

The structure of this report is very similar to that of the presentation: in the first part, important tools for the construction of the heterotic string are introduced, investigating the fundamental topic of modular invariance of the one-loop partition function; the second part deals then with the effective construction.

2 Modular Invariance

In the construction of the heterotic string, one fact will be of crucial importance: modular invariance of its partition function. Because of this property, one can restrict the possibilities for its gauge group up to two. The first part of this report is aimed to give an insight into this fundamental topic. Firstly, the concept of string partition function will be introduced, and the property of modular invariance will be explained. Secondly, the relation between the "genus-one" string partition function and the CFT partition function on a torus will be investigated. In this way, modular invariance of a string partition function can be transferred to a CFT partition function. This will give a convenient frame for the construction of the heterotic string.

The following is mainly based on [2], with relevant contributions from [4]. The reader who is not interested in the technical issues needed to understand modular invariance, can read the first and the last section of this part only. For simplicity, the discussion is restricted to bosonic coordinates.

2.1 The string partition function

The Polyakov path integral Interactions of closed strings can be heuristically represented by their joining and splitting. On the world-sheet level, the relevant surface is a two dimensional holed surface with boundary, where the latter represents in- and outgoing states.

The key observation is that by conformal invariance of the Polyakov action on the world-sheet, one can conformally map the boundaries to *punctures*. After the mapping, the quantum numbers of asymptotic states are implemented by operators inserted at some points. For example the world-sheet of a free propagating string, a cylinder, can be conformally mapped to the complex plane and by stereographic projection (which is conformal either) to the sphere. On the sphere, the asymptotic states (boundaries) are mapped to the two poles. This leads to the notion of *vertex operator*.

In the computation of string amplitudes one should consider correlations of vertex operators on two dimensional compact oriented surfaces without boundaries. Topologically, they are completely classified as spheres with g handles, where the number of holes is called the *genus* of the surface.

One should now not be completely surprised that n-point (n asymptotic states), g-loop amplitudes of closed oriented strings are usually computed as in the following formal path integral, called the *Polyakov path integral*:

$$\begin{aligned} A_n &= \sum_{g=0}^{\infty} A_n^{(g)} \\ &= \sum_{g=0}^{\infty} C_{\Sigma_g} \int \mathcal{D}h\mathcal{D}X^{\mu} \int d^2z_1...d^2z_n V_1(z_1,\bar{z_1})...V_n(z_n,\bar{z_n}) e^{-S[X,h]}. \end{aligned} \tag{1}$$

Here the sum runs over all topologies of the world-sheet, C_{Σ_g} are constants depending only on the world-sheet topology, h denote the metrics, X the embeddings from the world-sheet into the target space, and the V's are vertex operators. The $A_0^{(g)}$'s are called the *genus-g partition functions*. More details can be found in [3], [4].

Redundancy of the Polyakov path integral and modular invariance
As should be known (else, see for example [4]), the action S is invariant under conformal transformations and diffeomorphisms of the world-sheet. For that reason, the Polyakov path-integral as written above is redundant and highly divergent. In fact, one is integrating on infinitely many equivalent configurations.

To compensate this, one has to divide the integration measure by the volume of the symmetry groups generated by diffeomorphisms and Weyl rescalings:

$$\int \frac{\mathcal{D}h\mathcal{D}X}{\mathrm{Vol(Diff)Vol(Weyl)}}. \tag{2}$$

In other words, in the absence of anomalies the integration should be performed on a moduli space of metrics, in which metrics that can be transformed into each other via diffeomorphisms and Weyl rescalings are identified. Denoting the space of metrics on a compact two dimensional surface with genus g by $\mathcal{G}_g$, one defines the *moduli space* as the quotient:

$$\mathcal{M}_g = \frac{\mathcal{G}_g}{(\text{Diff} \times \text{Weyl})_g}. \tag{3}$$

Modular invariance is the invariance under the action of the so called *modular group*. In general Diff_g is not connected. The modular group is defined as the quotient of Diff_g with its component connected to the identity, Diff_0,

$$\text{Modular Group} = \frac{\text{Diff}_g}{\text{Diff}_0}. \tag{4}$$

By construction, once that the integration is restricted to the moduli space the string partition function must be modular invariant, since configurations related by diffeomorphisms are equivalent.

The next problem is to perform the restriction, *i.e.* to find an appropriate integration measure on the moduli space. To that end, one has to disentangle the integral over metrics into an integral over diffeomorphisms, an integral over Weyl rescalings and an integral over moduli. This procedure is sketched below.

2.2 The integration measure on the moduli space

The operator P and the dimension of the moduli space Under reparametrizations and Weyl rescalings the metric changes as (see for example [2])

$$\begin{aligned} \delta h_{\alpha\beta} &= \underbrace{-(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha)}_{\text{Diffeo.}} + \underbrace{2\Lambda h_{\alpha\beta}}_{\text{Weyl}} \\ &\equiv -(P\xi)_{\alpha\beta} + 2\tilde{\Lambda} h_{\alpha\beta}, \end{aligned}$$

where ∇ is the usual Christoffel connection, Λ is a function on the world-sheet coordinates to the real numbers and ξ is a vector. P is defined as the operator that maps vectors into symmetric traceless tensors according to

$$(P\xi)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - (\nabla_\gamma \xi^\gamma) h_{\alpha\beta},$$

whereas $2\tilde{\Lambda} = 2\Lambda - \nabla_\gamma \xi^\gamma$.

Once that a metric $h_{\alpha\beta}$ is fixed, not all other metrics can be generated via Weyl rescaling and diffeomorphimsms. These are the metrics which are reached by a change in the modular parameters. Explicitly, there are such metrics if the operator $P^\dagger$, which maps traceless symmetric tensors to vectors according to

$$(P^\dagger t)_\alpha = -2\nabla^\beta t_{\alpha\beta}$$

has zero modes. In fact, $(P^\dagger t_0)_\alpha = 0$ implies $(P\xi, t_0) = (\xi, P^\dagger t_0) = 0$ for any vector ξ. The dimension of the moduli space is therefore the number of zero modes of $P^\dagger$. The scalar product which was used is the canonical scalar product in the tangent space, which will be defined below.

The operator P and the conformal Killing group In the previous paragraph, changes in the moduli have been singled out from the possible changes of the metrics. The next step would be to find an *orthogonal* decomposition of these changes in order to obtain a suitable integration measure on the moduli space. However, there is a complication which already at this point can be discussed well.

If the diffeomorphisms and Weyl groups overlap, then some diffeomorphisms can be undone by a Weyl rescaling. In other words, there can be a subgroup of the diffeomorphisms and Weyl groups which leaves the metric invariant. This subgroup represent a symmetry that cannot be fixed by restricting the integration on the moduli space. Therefore, one has to take this group into account to avoid a further overcounting. It is called *conformal Killing group* (CKG).

Gladly, it is not hard to find the generators of the CKG, or *conformal Killing vectors* (CKVs). Consider the infinitesimal variation of the metric above. CKVs must leave the metric unchanged: this is the case for vectors ξ satisfying

$$(P\xi)_{\alpha\beta} = 0, \tag{5}$$

thus CKVs correspond to zero modes of P.

Complex coordinates and Riemannian surfaces It turns out to be very convenient stopping the previous investigations for a while, in order to introduce some useful language. In particular, this paragraph and the next few are intended to present concepts related to complex manifolds which will have great importance later. The discussion is based on both [2] and [4], which are recommended for futher details.

A compact manifold always admits a Riemannian metric. On a worldsheet of Minkowski signature, one can locally introduce coordinates such that the metric is of the form

$$ds^2 = 2e^{2\psi}((d\sigma^1)^2 - (d\sigma^0)^2). \tag{6}$$

This is the so called "conformal gauge". After a Wick rotation $\sigma^0 = -i\sigma^2$ and introducing complex coordinates $z = \sigma^1 + i\sigma^2$ the metric becomes (remember: locally!)

$$ds^2 = 2e^{2\psi} dz d\bar{z} \equiv 2h_{z\bar{z}} dz d\bar{z}. \tag{7}$$

Covering the manifold with patches in which the metric has the above form, one obtains a complex manifold, which in two dimensions has the name *Riemannian surface.*

In short, a Riemannian surface has by definition a set of overlapping patches with complex coordinates $\{z_m, \bar{z}_m\}$ in each of them (m being the patch index), and holomorphic transition functions $z_m = f_{mn}(z_n)$. In the present case, holomorphicity of the transition functions is guaranteed by (see (7))

$$ds^2 \propto dz_m d\bar{z}_m \ \forall m. \tag{8}$$

It follows from above that up to a Weyl rescaling (and a diffeomorphism, but this is already contained in its definition), for any two dimensional Riemannian manifold there is a Riemannian surface. Actually, something more is true: the two sets are in one-to-one correspondence. One can prove the remaining direction by taking the metric $dzd\bar{z}$ in every patch and smooth them in the overlaps by a partition of unity.

There are two ways of thinking about different Riemannian surfaces. Either one chooses for each Riemannian surface the flat metric $dzd\bar{z}$ in each patch and different Riemannian surfaces have different transition functions; or one chooses a fix coordinate system and different Riemannian surfaces have metrics of the form

$$ds^2 \propto |dz + \mu d\bar{z}|^2 \tag{9}$$

where $\mu = \mu_{\bar{z}}^{z}(z, \bar{z}) = h^{z\bar{z}} \delta h_{\bar{z}\bar{z}}$. $\mu_{\bar{z}}^{i,z} \equiv h^{z\bar{z}} \partial_i h_{\bar{z}\bar{z}}$ are called *Beltrami differentials.*

Covariant derivatives and tensor bundles Having a Riemannian surface one can define covariant derivatives. In the following, it will be always used the Levi-Civita connection.

The non-vanishing Christoffel symbols read (recall that $h_{z\bar{z}} = e^{2\psi}$ is the only non-vanishing metric component)

$$\begin{aligned} \Gamma_{zz}^{z} &= h^{zw} \frac{1}{2} \left(\partial_z h_{zw} + \partial_z h_{zw} - \partial_w h_{zz} \right) = \partial_z \psi, \\ \Gamma_{\bar{z}\bar{z}}^{\bar{z}} &= \partial_{\bar{z}} \psi. \end{aligned} \tag{10}$$

Covariant derivatives acts on tensors. Only holomorphic tensors will be treated here. These transform under analytic coordinate transformation $z \to f(z)$ as $T(z,\bar{z}) \to (\partial_z f(z))^n T(f(z), \bar{f}(\bar{z}))$. The space of tensors of rank n will be denoted by $\mathcal{T}^{(n)}$. It has a scalar product and norm defined as

$$\begin{aligned} (V^{(n)}, W^{(n)}) &= \int d^z \sqrt{h} (h^{z\bar{z}})^n V^{(n)*} W^{(n)} \\ ||V^{(n)}||^2 &= (V^{(n)}, V^{(n)}). \end{aligned} \tag{11}$$

The covariant derivatives act as follows:

$$\begin{aligned} &\nabla_z^{(n)} : \mathcal{T}^{(n)} \to \mathcal{T}^{(n+1)} \\ &\nabla_z^{(n)} T^{(n)} = (\partial - 2n\partial\phi) T^{(n)}; \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\nabla^z_{(n)} : \mathcal{T}^{(n)} \to \mathcal{T}^{(n-1)} \\ &\nabla^z_{(n)} T^{(n)} = h^{z\bar{z}} \nabla_{\bar{z}} T^{(n)} = h^{z\bar{z}} \bar{\partial} T^{(n)}. \end{aligned} \tag{13}$$

Moreover, one has the relation $(\nabla_z^{(n)})^\dagger = -\nabla^z_{(n+1)}$. Setting $n = 1$, one should note $(PV)_{zz} = 2\nabla_z^{(1)} V_z$ and $(PV)^{zz} = 2\nabla^z_{(2)} V^z$ where P is the operator defined above.

Finally, for later purposes it should be mentioned that the Riemann curvature tensor has only one independent component

$$R_{z\bar{z}z\bar{z}} = 2e^{2\psi} \partial_z \partial_{\bar{z}} \psi, \tag{14}$$

and that therefore the Ricci scalar curvature is

$$R = -4e^{-2\psi} \partial_z \partial_{\bar{z}} \psi. \tag{15}$$

Quadratic differentials, Beltrami differentials and CKVs This paragraph presents all the objects relevant for the decomposition of the tangent space of metrics in the complex language.

Zero modes of the adjoint of $\nabla_z^{(+1)}$, that is $-\nabla^z_{(+2)}$, are called *quadratic differentials*. Satisfying

$$h_{z\bar{z}} \nabla^z_{(+2)} \phi_{zz} = \partial_{\bar{z}} \phi^i_{zz} = 0 \tag{16}$$

they are global analytic tensors of rank 2. Moreover, they have a natural pairing with Beltrami differentials

$$(\mu^i, \phi^j) = \int d^2 z \mu^{iz}_{\bar{z}} \phi^j_{zz}. \tag{17}$$

The conformal Killing vectors, which were defined to be the zero modes of P, can now be seen as $\mathcal{T}^{(1)}$ tensors spanning the kernel of $\nabla^{(+1)}_z$

$$\nabla^{(+1)}_z V_z = h_{z\bar{z}} \partial_z V^{\bar{z}}. \tag{18}$$

They are globally defined vector fields.

The decomposition All the concepts which are needed to decompose the space of metrics have been given now, and it is time to proceed. Start with a metric $\propto dzd\bar{z}$. To the end of getting an *orthogonal* decomposition, one needs to project the variation of the moduli parameters (Beltrami differentials) on the space spanned by quadratic differentials, which is the moduli space. Denoting the projector by $\Pi = \sum_{ij} |\phi_i)(\phi_i, \phi_j)(\phi_j|$ and defining $M_{ij} = (\phi_i, \phi_j)$, one finds:

$$\begin{aligned} \delta h_{z\bar{z}} &= \Lambda h_{z\bar{z}}, \\ \delta h_{zz} &= \nabla^{(+1)}_z \xi_z + \sum_{ijk} \phi^j_{zz} M^{-1}_{jk} (\phi^k, \mu^i) \delta\tau^i. \end{aligned} \tag{19}$$

Introducing the relative Jacobian, the integration over metrics becomes therefore:

$$\begin{aligned} \int_{\mathcal{G}_g} \prod_i d\tau_i^2 \int \frac{\mathcal{D}X\mathcal{D}'\xi\mathcal{D}\Lambda}{\text{Vol(Diff)Vol(Weyl)}} \frac{\det(\phi,\mu)\det(\mu,\phi)}{\det(\phi,\phi)} \times \\ \times \det{}' \nabla^{(+1)}_z \det{}' \nabla^z_{(-1)}. \end{aligned} \tag{20}$$

Here the prime means that the integration should not be performed on conformal Killing vectors, whose contribution is already taken into account in Λ.

Now, the inverse volume of the diffeomorphism not connected to the identity can be cancelled by restricting the integration on the moduli space. Moreover, one can decompose the volume of diffeomorphisms connected to the identity vol(Diff)_0 as

$$\text{vol(Diff)}_0 = \text{vol}(\text{Diff}^*_0)\text{vol(CKG)}. \tag{21}$$

In the absence of conformal anomaly, $\mathcal{D}\xi'$ and $\mathcal{D}\Lambda$ can be cancelled against $\mathrm{vol}(\mathrm{Diff}_0^*)\mathrm{vol}(\mathrm{Weyl})$. The final result is

$$\int_{\mathcal{M}_g} \prod_i d\tau_i^2 \int \frac{\mathcal{D}X}{\mathrm{Vol(CKG)}} \frac{\det(\phi,\mu)\det(\mu,\phi)}{\det(\phi,\phi)} \det{}' \nabla_z^{(+1)} \det{}' \nabla^z_{(-1)}. \tag{22}$$

2.3 The torus

From now on, the relevant surface will be the torus, that is $g = 1$, and the relevant partition function $A_0^{(1)}$. The moduli space and the modular group of the torus have already been exposed in a previous talk of this proseminar [5]. Recall that there is only one complex modular parameter τ, and that the moduli space is:

$$\mathcal{M}_1 = \mathbb{H}/\{\text{action of } \mathrm{PSL}_2(\mathbb{Z})\} \tag{23}$$

where $\mathbb{H}$ is the upper complex half plane without the real line, and the action of $\mathrm{PSL}_2(\mathbb{Z})$ is generated by

$$\begin{aligned} T &: \tau \to \tau + 1 \\ S &: \tau \to -\frac{1}{\tau}. \end{aligned} \tag{24}$$

The moduli space can be seen as a "fundamental region" on the upper complex half plane

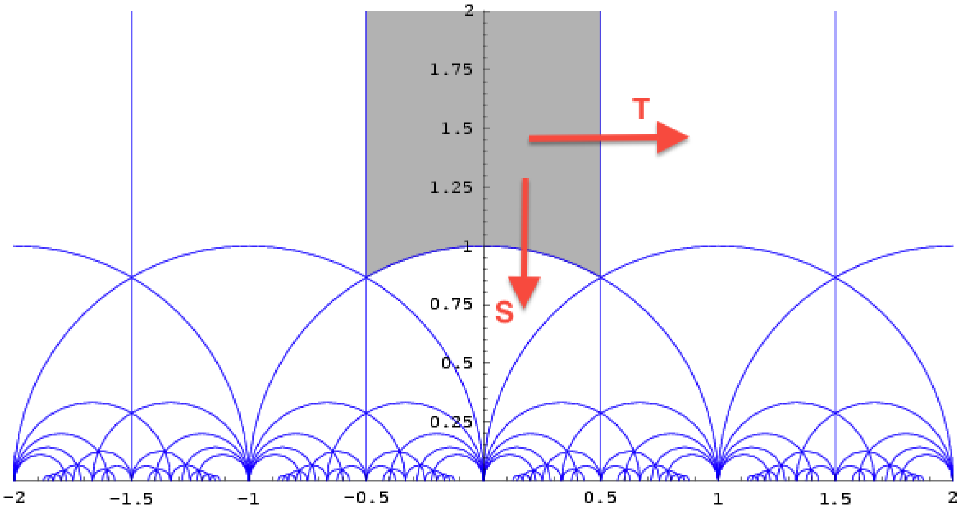
$$\mathcal{F} = \{z \in \mathbb{H} | |\mathrm{Re}(z)|^2 \leq \frac{1}{2},\ |z| \geq 1\}. \tag{25}$$

The fundamental region $\mathcal{F}$ and the action of the two generators are depicted in Figure 1.

The task is now to compute the beltrami and quadratic differentials, and to find the conformal K Killing vectors.

Beltrami-, quadratic differentials and conformal Killing vectors of the torus The torus can be parametrized with two real variables $0 \leq \xi^1, \xi^2 \leq 1$ such that the complex coordinates become $z = \xi^1 + \tau\xi^2$, where τ is the moduli parameter. Setting the metric $ds^2 = |dz^2|$, and changing $\tau \to \tau + \delta\tau$ keeping the parametrization fixed (the previous "second way" of thinking about a Riemannian surface), one finds $ds^2 \to\sim |dz + \delta\tau \frac{i}{2\mathrm{Im}(\tau)} d\bar{z}|^2$. Therefore,

Figure 1: The fundamental region $\mathbb{F}$ of the torus and the action of the modular group



$$\mu_{\bar{z}}^{z}(z,\bar{z}) = \frac{i}{2\text{Im}(\tau)}.$$

The conformal Killing vectors can be found by means of the Ricci identity:

$$\left(\nabla_{(n+1)}^{z}\nabla_{z}^{(n)} - \nabla_{z}^{(n-1)}\nabla_{(n)}^{z}\right) = \frac{1}{2}nR, \tag{26}$$

where R is the Ricci curvature scalar, which is zero in the case of the torus since it always admits a globally flat metric $dzd\bar{z}$, which will be used in the following. In fact, let $V^{(1)}$ be conformal Killing vectors, then

$$\begin{aligned} 0 &= (\nabla_{z}^{(1)}V^{(1)}, \nabla_{z}^{(1)}V^{(1)}) \\ &= -(V^{(1)}, \nabla_{(n+1)}^{z}\nabla_{z}^{(1)}V^{(1)}) \\ &= \frac{1}{2}\left(||\nabla_{z}^{(1)}V^{(1)}|| + ||\nabla_{(1)}^{z}V^{(1)}|| - \frac{1}{2}(V^{(1)}, RV^{(1)})\right) \\ &= \frac{1}{2}\left(||\nabla_{z}^{(1)}V^{(1)}|| + ||\nabla_{(1)}^{z}V^{(1)}||\right). \end{aligned} \tag{27}$$

In the first line it has been used that CKVs are zero modes of the covariant derivative, in the third line the Ricci equation has been used. The last line is a consequence of the vanishing of the ricci curvature in the case of the torus. This implies

$$\partial_z V^{(1)} = \partial_{\bar{z}} V^{(1)} = 0 \tag{28}$$

or in words, that CKVs are constants. In order to compute the volume of the CKG, one notes that CKVs are spanned by the constants fields ∂_{ξ^1} and ∂_{ξ^2}, which generates shifts $\xi^\alpha \to \xi^\alpha + a^\alpha$ with $0 \le a^\alpha \le 1$. The metric on the space of the CKVs is $g_{ij} = \int \sqrt{h} h_{\alpha\beta} V_i^\alpha V_j^\beta$, hence

$$\mathrm{Vol(CKG)} = \int \sqrt{\det(g)} da^1 da^2 = \mathrm{Im}(\tau). \tag{29}$$

Integration on the moduli Finally, one obtains for the genus-one partition function

$$A_0^{(1)} \propto \int_{\mathcal{F}} d\tau \int \frac{\mathcal{D}X}{\mathrm{Im}(\tau)} \frac{1}{\mathrm{Im}(\tau)^2} \det{}' \nabla_z^{(+1)} \det{}' \nabla^z_{(-1)} e^{-S[X,\tau]}. \tag{30}$$

This can be evaluated further (see [2]) to get

$$\int_{\mathcal{F}} d\tau \int \frac{\mathcal{D}X}{\mathrm{Im}(\tau)} \frac{1}{\mathrm{Im}(\tau)^2} \mathrm{Im}(\tau)^{13} \det{}'(\Box) \det{}' \nabla_z^{(+1)} \det{}' \nabla^z_{(-1)}. \tag{31}$$

where the $\Box$ is the Laplace operator.

Equivalently, one could derive the right measure in the Faddeev-Popov formalism, where the symmetries are fixed by means of ghost insertions. The result must be clearly the same. Quadratic differentials and conformal Killing vectors represent ghost zero-modes. One can rewrite Eq. 31 as

$$A_0^{(1)} \propto \int_{\mathcal{F}} d\tau \frac{1}{\mathrm{Im}(\tau)} \frac{1}{\mathrm{Im}(\tau)^2} \mathrm{Im}(\tau)^{13} \int \mathcal{D}' X \mathcal{D}' c \mathcal{D}' b e^{-S[X,\tau] - S[b,c,\tau]} \tag{32}$$

where c, b are ghost fields and the prime means that one should not integrate on zero modes.

Modular invariance of the integration measure An expression for the genus-one partition function, Eq. 32, has been finally obtained. The result is an integral on the moduli space, with a measure carrying some factors in the order of $\mathrm{Im}(\tau)$. The evaluation of the integrand (a path integral as in the form above) will be the task of the next subsection. Before, one should note that some important information can already be extracted at this point. In fact remember from 2.1 that the one-loop partition function must be modular invariant. Thus, investigating the behavor of the measure under modular transformations, one can already fix in which terms the integrand must transform. But the former is very easy, and will be done in a moment. In the second part of the report, the resulting conditions for the integrand will translate in a restriction on the possible gauge groups of the heterotic string.

Let therefore consider an arbitrary modular transformation, that has the form (see the generators in Eq. 24)

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{33}$$

where a b c d are integers (nothing to do with ghost fields!) satisfying $ad - cb = 1$. Observe:

$$\begin{aligned} d^2\tau &\to |c\tau + d|^{-4} d^2\tau \\ \mathrm{Im}(\tau) &\to |c\tau + d|^{-2} \mathrm{Im}(\tau). \end{aligned} \tag{34}$$

This means that the modular invariant integration measure is

$$\frac{d^2\tau}{\mathrm{Im}(\tau)^2}. \tag{35}$$

Further factors of $\mathrm{Im}(\tau)$ have to be compensated by the integrand.

2.4 String partition function, CFT partition function and modular invariance

The remaining integrand as a CFT partition function As already mentioned, it remains to evaluate the integrand, which is a path-integral on the fields. There is one important fact that one should keep in mind in this context: for conformal field theories, the partition function corresponds to the generating functional in quantum field theory which is expected, since the thermodynamic expression can be deduced from an Euclidean quantum field theory with time compactified on a circle of radius $R = \frac{1}{T}$ (T: temperature).

Given this observation, one should note that the path-integral on the fields is something that has already been computed in a previous session of the proseminar, namely [5]. It corresponds in fact to the CFT partition function of the torus

$$Z(\tau_1, \tau_2) = \mathrm{Tr}_{\mathcal{H}}(e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P}), \tag{36}$$

where H, P generate translations in τ_1, τ_2. In a nutshell, the motivation for this formula is that setting $\tau_1 = 0$ one sees that the partition function counts the number of states propagating around τ_2 weighted by the factor

$e^{-2\pi\tau_2 H}$. The factor $e^{2\pi\tau_1 P}$ comes out because on the torus, time translations of length τ_2 end up on a point displaced in space by τ_1 (the torus is "twisted").

The fermionic and bosonic partition functions have already been treated in [5], and cited in talk [7]. Their derivation goes clearly away from the purposes of this report. They will be recalled and used in the next part, where they are needed for constructing the heterotic string partition function. There, the partition function for a compactified boson will also be introduced.

The real problem is represented by the contributions of the ghost-fields. One finds that (see [4]) they usually cancel out two sets of uncompactified bosons/fermions (except zero-modes), and that therefore the string one-loop amplitude can be written as

$$A_0^{(1)} \propto \int \underbrace{\frac{d^2\tau}{\mathrm{Im}(\tau)^2}}_{\text{Modular Invariant}} \underbrace{\frac{1}{\mathrm{Im}(\tau)^{-\frac{D}{2}+1}} Z(\tau,\bar{\tau})}_{\tilde{Z}} . \tag{37}$$

D is the number of non-compact dimensions, and $Z(\tau,\bar{\tau})$ counts the states in the theory without zero modes and without two sets of fermions/bosons. One should note that this is equivalent to working in light-cone gauge since the light-cone coordinates have no oscillator contributions.

Modular invariance The final result, which will be used to construct the heterotic string, is the following

> *The torus CFT partition function $\tilde{Z}(\tau,\bar{\tau})$ of any string theory as defined above must be modular invariant.*

3 Construction of the heterotic string

Having acquired the tools which have been presented in the previous part, the construction of the heterotic string results to be not so difficult. Here it is structured as follows. At the beginning, a quick introduction explains the motivations which lead to the heterotic string. Next, its constituent parts are exposed. The partition function $\tilde{Z}$ as defined above can be then easily computed. The requirement of its modular invariance is furher translated into a restriction for the gauge group of the heterotic string. It will result that there are only two possibities: one is the famous group $E_8 \times E_8$, the other $SO(32)$. The reference is mainly [2].

3.1 Introduction

Previous talks already presented some good features of the superstring (see [7]). In particular, after GSO projection –required by modular invariance– the superstring's spectrum contains no tachyons and has a graviton. However, no interaction in space-time has been considered until now. In the standard model language, interactions are the same as gauge symmetries. More specifically, to reproduce the standard model one has to introduce a space-time gauge symmetry $U(1) \times SU(2) \times SU(3)$.

The heterotic string attempts to add a gauge symmetry by mixing the left-moving sector of the bosonic string with the right-moving sector of the superstring ("hybris"), and compactifying the extra bosonic coordinates needed to compensate the anomaly on a torus (remember that 16 more coordinates are needed for the bosonic part, 26 in total, see [2]). The compactified bosons will take the role of gauge bosons, and the result will be a 10 dimensional string theory with a $E_8 \times E_8$ or $SO(32)$ gauge symmetry, much larger than the standard model's one.

3.2 The two sectors

As explained above, the heterotic string combines the left-moving sector of the 26-dimensional bosonic string with the right-moving sector of the 10-dimensional superstring. As such it is a string theory in 10 dimensions, with 16 internal (bosonic) degrees of freedom compactified on a torus. Therefore, it deals with the following fields:

1. Left-moving coordinates

    - 10 uncompactified bosonic fields $X^{\mu}_{L}(\tau + \sigma)$, $(\mu = 0, ..., 9)$

- 16 internal bosons $X^I_L(\tau+\sigma)$ $(I = 1, ..., 16)$ living on a 16-dimensional torus

2. Right-moving coordinates

- 10 uncompactified bosons $X^\mu_R(\tau - \sigma)$, $(\mu = 0, ..., 9)$ with their
- two-dimensional fermionic superpartners $\Psi^\mu_R(\tau - \sigma)$.

Moreover, one gets left- and right-moving reparametrization ghosts $b, c, \bar{b}, \bar{c}$ and right-moving superconformal ghosts β, γ. Ghost contributions will be only taken into account when computing the partition function.
Note that $X^\mu_L(\tau+\sigma)$, $X^\mu_R(\tau-\sigma)$ have common center of mass and space-time momentum coordinates, with continuous spectra. One will see that center of mass positions and momenta of the compactified coordinates X^I are instead subjected to discretizing conditions.

Independence of the left- and right moving sectors (bosonic string)
Recall that $X^\mu_R(\tau-\sigma)+X^\mu_L(\tau+\sigma)$ is the general solution of the closed string equation of motion $(\partial^2_\sigma - \partial^2_\tau)X^\mu = 0$ with periodicity condition $X^\mu(\sigma, \tau) = X^\mu(\sigma + l, \tau)$. Coordinates of both sectors must therefore be of the form:

$$X^\mu_R(\tau - \sigma) = \frac{1}{2}(x^\mu) + \frac{\pi\alpha'}{l}p^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-\frac{2\pi}{l}in(\tau-\sigma)} \tag{38}$$

$$X^\mu_L(\tau + \sigma) = \frac{1}{2}(x^\mu) + \frac{\pi\alpha'}{l}p^\mu(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^\mu_n e^{-\frac{2\pi}{l}in(\tau+\sigma)}. \tag{39}$$

In the frame of canonical quantization, the above (promoted) operators satisfy

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{40}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu,0} \tag{41}$$

$$[\bar{\alpha}^\mu_m, \alpha^\nu_n] = 0. \tag{42}$$

Note that necessarily having common center of mass position and momentum, the two sectors are not manifestly independent. In fact, calculating the correlator between them, one gets

$$\langle X^\mu_R(z)X^\nu_L(\bar{w})\rangle = -\frac{1}{4}\alpha'\eta^{\mu\nu}\ln z \tag{43}$$

$$\langle X^\mu_L(\bar{z})X^\nu_R(w)\rangle = -\frac{1}{4}\alpha'\eta^{\mu\nu}\ln \bar{z}, \tag{44}$$

where cylinder coordinates $(z, \bar{z}) = (e^{2\pi i(\tau-\sigma)/l}, e^{2\pi i(\tau+\sigma)/l})$ are used for convenience. Nevertheless, only propagators for $X = X_R + X_L$ matter. One sees that redefining commutators $[x_R^\mu, p_L^\nu] = 0$, identical correlators (and hence, propagators) are obtained:

$$\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\rangle = -\frac{1}{2}\alpha' \eta^{\mu\nu} \ln(z-w)(\bar{z}-\bar{w}). \tag{45}$$

The zero-mode dependence of X_R and X_L is therefore irrelevant.

Discretized momenta Momenta of the compactified coordinates are discretized. To see this, recall the case of one dimension first, which has been treated in talk [6]. Letting x and p be the center of mass position and momentum of the coordinate respectively, single-valuedness of the wave function $\exp(xp)$ provides

$$p^i = \frac{M}{R},\ M \in \mathbb{Z}, \tag{46}$$

where R is the radius of the compactified string.

If there are more dimensions to compacify, one has to identify points of a lattice rather than points of a single line,

$$X^I \sim X^I + 2\pi \sum_{i=1}^{D} n^i e_i^I = X^I + 2\pi L^I,\ n_i \in \mathbb{Z}.$$

Here the $\{e_i\}_{i=1...D}$ are basis vectors of a lattice D dimensional lattice Λ. For the heterotic string $D = 16$, as explained above. Now, similarly to the one dimensional case one has to require single-valuedness of the wave function $\exp(ix^i p_i)$, where $\{x^i, p_i\}_{\{i=1,...16\}}$ are the center of mass positions and momenta of the 16 coordinates. This implies directly that the momenta of the compactified bosons must be vectors of the 16-dimensional lattice $\Gamma_{16} = \Lambda^*$ dual to Λ.

Finally, a piece of notation: letting for simplicity $e_i^I, I = 1, ..., 16$ be the basis vectors of the momenta lattice (the only one which will be used later on), the corresponding metric will be denoted, as usual, $g_{ij} \equiv \sum_{I=1}^{16} e_i^I e_j^I$.

3.3 Modular invariance and the gauge group of the heterotic string

One-loop partition function The task of this paragraph is to compute the one-loop partition function of the heterotic string, with the right amount of contributions, by means of Eq. 36. Recall therefore from the end of the first part that two sets of noncompact bosons/fermions have to be cancelled in order to compensate the ghosts' contributions. Moreover, recall the Virasoro characters for fermions and bosons, whose derivations can be found in [5], [2]:

$$\chi_{8-\text{fermions}}(\tau) = \frac{1}{2}\frac{1}{|\eta(\tau)|^4}\left(\theta(\tau)_3^4 - \theta(\tau)_4^4 - \theta(\tau)_2^4\right)$$

$$\chi_{\text{n}-\text{bosons}}(\tau) = \left(\frac{1}{|\eta(\tau)|}\right)^n.$$

Here $\eta(\tau)$, $\theta_1 \equiv \theta_{1/2}^{1/2}$, $\theta_2 \equiv \theta_0^{1/2}$, $\theta_3 \equiv \theta_0^0$, $\theta_4 \equiv \theta_{1/2}^0$ are the Dedekind eta-function and the theta-functions, defined as ($q = e^{2\pi i\tau}$)

$$\begin{aligned} \eta(\tau) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \\ \theta_\beta^\alpha(\tau) &= \sum_{n\in\mathbb{Z}} e^{i\pi(n+\alpha)^2\tau + 2\pi i(n+\alpha)\beta}. \end{aligned} \tag{47}$$

In addition, for the heterotic string one needs the character of compactified bosons. It is not difficult to derive. The trace of Eq. 36 simply counts the states on the torus as follows

$$\chi_{16-\text{comp.bosons}}(\tau) = \frac{1}{|\eta(\tau)|}^{16}\sum_{\mathbf{p}_L\in\Gamma_{16}} q^{\frac{1}{2}\mathbf{p}_L^2}.$$

Hence, taking all contributions the heterotic string partition function without zero modes reads

$$\begin{aligned} Z_{\text{het}}(\tau,\bar\tau) &= \chi_{8-\text{fermions}}(\tau)\chi_{8-\text{bosons}}(\tau)\chi_{8-\text{bosons}}(\bar\tau)\chi_{16-\text{comp.bosons}}(\bar\tau) \\ &= \frac{1}{|\eta(\tau)|^4}\left(\theta(\tau)_3^4 - \theta(\tau)_4^4 - \theta(\tau)_2^4\right)\left(\frac{1}{|\eta(\tau)|}\right)^8\left(\frac{1}{|\eta(\bar\tau)|}\right)^8 \\ &\times \frac{1}{|\eta(\bar\tau)|^{16}}\sum_{\mathbf{p}_L\in\Gamma_{16}}\bar q^{\frac{1}{2}\mathbf{p}_L^2} \\ &= \left(\frac{1}{[\eta(\bar\tau)]^{24}}\sum_{\mathbf{p}_L\in\Gamma_{16}}\bar q^{\frac{1}{2}\mathbf{p}_L^2}\right)\left(\frac{1}{[\eta(\tau)]^{12}}\left(\theta_3^4(\tau) - \theta_4^4(\tau) - \theta_2^4(\tau)\right)\right). \end{aligned} \tag{48}$$

The result with zero-modes is finally

$$\tilde{Z}_{het}(\tau,\bar{\tau}) = \frac{1}{(\mathrm{Im}\tau)^4}\left(\frac{1}{[\eta(\bar{\tau})]^{24}}\sum_{\mathbf{p}_L\in\Gamma_{16}}\bar{q}^{\frac{1}{2}\mathbf{p}_L^2}\right)\times \\ \times\left(\frac{1}{[\eta(\tau)]^{12}}\left(\theta_3^4(\tau)-\theta_4^4(\tau)-\theta_2^4(\tau)\right)\right). \tag{49}$$

Action of the generators of the modular group In the previous paragraph, the one-loop partition function of the heterotic string has been quickly computed. At this point, however, the lattice Γ_{16} of the compactified momenta is complely arbitrary. The next step corresponds to investigate the effect of the generators of the modular group on the partition function. By imposing modular invariance, this leads to some constraints for the lattice under the form of transformation rules.

In the first part of this report it has been recalled that the generators are:

$$\begin{aligned} T &: \tau \to \tau + 1 \\ S &: \tau \to -\frac{1}{\tau}. \end{aligned} \tag{50}$$

Transformations of the eta-function, the theta-functions, and $\mathrm{Im}(\tau)$ under the action of the modular group are known:

	$T:\tau\to\tau+1$	$S:\tau\to-\frac{1}{\tau}$
$\eta(\tau)$	$e^{\frac{i\pi}{12}}\eta(\tau)$	$\sqrt{-i\tau}\eta(\tau)$
$\theta_2(\tau)$	$e^{\frac{\pi i}{4}}\theta_2(\tau)$	$\sqrt{-i\tau}\theta_4(\tau)$
$\theta_3(\tau)$	$\theta_4(\tau)$	$\sqrt{-i\tau}\theta_3(\tau)$
$\theta_4(\tau)$	$\theta_3(\tau)$	$\sqrt{-i\tau}\theta_2(\tau)$
$\mathrm{Im}(\tau)$	$\mathrm{Im}(\tau)$	$\frac{\mathrm{Im}(\tau)}{\lvert\tau\rvert}$

The only factor whose transformation in yet unknown, is indeed the lattice (or "soliton", since states in it are topologically stable) sum

$$P(\tau) = \sum_{\mathbf{p}_L\in\Gamma_{16}}\bar{q}^{\frac{1}{2}\mathbf{p}_L^2} \tag{51}$$

where the summation runs over all vectors of the lattice. After having implemented the above transformations into the partition function Eq. 49, one easily sees that modular invariance requries

$$P(\tau+1) = P(\tau) \tag{52}$$

$$P(-\frac{1}{\tau}) = \tau^8 P(\tau). \tag{53}$$

Transformation of the soliton sum under T The constraints above Eq. 52 and 53 can be translated into interesting properties of the lattice. Starting with the first one

$$P(\tau+1) = \sum_{\mathbf{p}_L \in \Gamma_{16}} \bar{q}^{\frac{1}{2}\mathbf{p}_L^2} e^{\pi i \mathbf{p}_L^2} = P(\tau) \Rightarrow \mathbf{p}_L^2 \in 2\mathbb{Z} \tag{54}$$

which means that Γ_{16} must be an *even* lattice. Now, $\mathbf{p}_L^2 = \sum_i p^i g_{ij}^j = \sum_i (p^i)^2 g_{ii} + 2\sum_{i<j} p^i g_{ij} p^j$ for $p^i, p^j \in \mathbb{Z}$, and hence diagonal elements of the metric must be even integers, $g_{ii} \in 2\mathbb{Z}$.

Transformation of the soliton sum under S The investigation of the second equation is more subtle, but a simple trick makes things easy. Start with the transformed soliton sum, which reads

$$P(-\frac{1}{\tau}) = \sum_{\mathbf{p}_L \in \Gamma_{16}} e^{-\pi i \frac{1}{\tau}\mathbf{p}_L^2}. \tag{55}$$

Defining now

$$F(\mathbf{x}) = \sum_{\mathbf{p}_L \in \Gamma_{16}} e^{-\pi i \frac{1}{\tau}(\mathbf{p}_L + \mathbf{x})^2}; \tag{56}$$

since the sum runs over all vectors, one has

$$F(\mathbf{x}+\mathbf{p}) = F(\mathbf{x}) \ \forall \mathbf{p} \in \Gamma_{16}. \tag{57}$$

Owning this periodicity, F can be then expanded in a Fourier series:

$$F(\mathbf{x}) = \sum_{\mathbf{q} \in \Gamma_{16}^*} e^{2\pi i \mathbf{x}\cdot\mathbf{q}} F^*(\mathbf{q}) \tag{58}$$

where

$$F^*(\mathbf{q}) = \frac{1}{\mathrm{vol}(\Gamma_{16})} \int_{\text{unit cell}} d^n y e^{-2\pi i \mathbf{y}\cdot\mathbf{q}} F(\mathbf{y}). \tag{59}$$

Inserting Eq. 56 in Eq. 58, Eq. 59 one obtains

$$
\begin{aligned}
&\sum_{\mathbf{p}_L\in\Gamma_{16}} e^{\pi i\frac{1}{\tau}(\mathbf{p}_L+\mathbf{x})^2} = \\
&= \sum_{\mathbf{q}\in\Gamma^*_{16}} e^{2\pi i\mathbf{x}\cdot\mathbf{q}}\frac{1}{\mathrm{vol}(\Gamma_{16})}\int_{\text{unit cell}} d^n y e^{-2\pi i\mathbf{y}\cdot\mathbf{q}} \sum_{\mathbf{p}_L\in\Gamma_{16}} e^{\pi i\frac{1}{\tau}(\mathbf{p}_L+\mathbf{x})^2} \\
&= \frac{1}{\mathrm{vol}(\Gamma_{16})}\sum_{\mathbf{q}\in\Gamma^*_{16}} e^{2\pi i\mathbf{x}\cdot\mathbf{q}} \underbrace{\sum_{\mathbf{p}_L\in\Gamma_{16}}\int_{\text{unit cell}}}_{=\int_{\mathbb{R}^n} d^n y} d^n y\, e^{-2\pi i(\mathbf{y}+\mathbf{p}_L)\cdot\mathbf{q}} e^{\pi i\frac{1}{\tau}(\mathbf{p}_L+\mathbf{x})^2} \\
&= \frac{1}{\mathrm{vol}(\Gamma_{16})}\sum_{\mathbf{q}\in\Gamma^*_{16}} e^{2\pi i\mathbf{x}\cdot\mathbf{q}}\int_{\mathbb{R}^n} d^n y e^{-\pi i(\mathbf{y})^T\frac{1}{\tau}\mathrm{id}_{16\times16}(\mathbf{y})-2\pi i(\mathbf{y}+\mathbf{p}_L)\cdot\mathbf{q}} \\
&= \frac{1}{\mathrm{vol}(\Gamma_{16})\sqrt{\det(\frac{1}{\tau}\mathrm{id}_{16\times16})}}\sum_{\mathbf{q}\in\Gamma^*_{16}} e^{2\pi i\mathbf{x}\cdot\mathbf{q}} e^{-2\pi i\mathbf{q}^2}.
\end{aligned}
\tag{60}
$$

In the second line it has been used that $e^{-2\pi i(\mathbf{y}+\mathbf{p}_L)\cdot\mathbf{q}} = e^{-2\pi i\mathbf{y}\cdot\mathbf{q}}$ since $\mathbf{q}\cdot\mathbf{p}_L\in\mathbb{Z}$, and in the last step a usual Gaussian integral has been performed. Hence, evaluating at $\mathbf{x}=0$, one finally obtaines

$$
P(-\frac{1}{\tau}) = \frac{\tau^8}{\mathrm{vol}(\Gamma_{16})}\sum_{\mathbf{q}\in\Gamma^*_{16}} q^{\frac{1}{2}\mathbf{q}^2}. \tag{61}
$$

In other words, Γ_{16} must be a *self-dual* lattice! In fact

$$
\Gamma_{16} = (\Gamma_{16})^* \tag{62}
$$

implies $\det g = 1$, $\mathrm{vol}(\Gamma) = \mathrm{vol}(\Gamma)^* = 1$.

Gauge group of the heterotic string By the above, modular invariance of the one-loop partition function requires that momenta of the internal dimensions must be elements of an even, self-dual Euclidean lattice. These are very special. In 16 dimensions, one can show that the only lattices of this kind are the direct product lattice $\Gamma_{E_8}\times\Gamma_{E_8}$, where Γ_{E_8} is the root lattice of the exceptional simple lie group E_8, and $\Gamma_{D_{16}}$, the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$. The proof of this fact goes away from the purposes of this report, and will not be presented here.

Both the root lattice of $E_8\times E_8$ and the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$ contain 480 vectors of $(\text{length})^2 = 2$ which are the roots of $E_8\times E_8$ and $SO(32)$ respectively. This implies that the gauge group of the heterotic string is either $E_8\times E_8$ or $SO(32)$. The two lattices, their respective groups and some related facts are the object of the next subsection.

3.4 $\Gamma_{E_8} \times \Gamma_{E_8}$ and $\Gamma_{D_{16}}$

Modular invariance has singled out two possible lattices for the compactified momenta. These are lattices related to the weights and roots of some lie algebras. The aim of the following paragraphs is to give a look at these interesting objects.

The root lattice Recall (see [8]) that given a choice of a Cartan-Weyl basis, that is a maximal set of Hermitian commuting generators H^I, the remaining generators can be diagonalized with respect to its elements:

$$[H^I, E^\alpha] = \alpha^I E^\alpha. \tag{63}$$

The vectors α^I are called *roots*, and taking arbitrary integer linear combinations of them one gets the *root lattice* Λ_R.

The weight lattice The weight lattice is constructed in the following way. States which transform into a specific representation of any Lie algebra can be denoted by

$$|\mathbf{m}_l, D\rangle \tag{64}$$

where l runs from 1 to D, the dimension of the representation. These are eigenstates of the Cartan subalgebra generators

$$H^I\,|c, D\rangle = m_l^I\,|\mathbf{m}_l, D\rangle\,. \tag{65}$$

The m_l, are called the *weight vectors*. Observe that it follows directly from the definition of roots that they correspond to the weights of the adjoint representation. Arbitrary linear integer combinations of weight lattice form the *weight lattice*.

A lie algebra is called simply laced if all its root vectors have the same length, which can be normalized to 2. For simply laced groups one has that $\alpha_i \cdot \mathbf{m} \in \mathbb{Z}$ for every root and weight, and if $\beta \cdot \mathbf{m} \in \mathbb{Z}\ \forall \mathbf{m}$, then $\beta \in \Lambda_R$. Summarizing, to the ends of this report one should retain the following:

- $\Lambda_R \subset \Lambda_w$ (since roots are the weights of the adjoint representation);
- $\Lambda_R = \Lambda_w^*$
- $\text{vol}(\Lambda) = \text{vol}(\Lambda^*)^{-1} \Rightarrow \text{vol}(\Lambda_R) = \text{vol}(\Lambda_w)^{-1}$.

The Lie-algebra lattice There is one further interesting lattice related to Lie algebras, which is simply called "Lie-algebra lattice". In order to construct it, first note that weight vectors of a representation differ by root vectors. In fact, the weights can be constructed from the highest weight by acting with the "lowering operators" E_α by Eq. 63. Now, for irreducible representations one can define conjugacy classes: if weight vectors of two representations differ by root vectors only, then the two representations lie in the same conjugacy class. One obtaines therefore the coset decomposition

$$\Lambda_w = \Lambda_R \oplus (\Lambda_R + \mathbf{m}_2) \oplus ... \oplus ...(\Lambda_R + \mathbf{m}_{N_C}) \tag{66}$$

where he vectors $\{\mathbf{m}_i\}_{i=1...N_c}$, N_c being the number of conjugacy classes, are representative for the conjugacy class. Usually the highest weight of the lowest dimensional representation is taken as representative.

Now, if it is the case that closeness under addition of vectors is provided, one can take only some of the conjugacy classes to form a lattice. In this way, one gets a so called *Lie-algebra lattice*.

The root lattice of E_8 After the previous general discussion, it is time to consider the specific cases of the lattices and groups (or better, algebras) relevant for the heterotic string, starting with the root lattice of E_8. First of all, E_8 is an exceptional Lie algebra[1] which is simply laced. As any textbook about Lie algebras explains, it is usually constructed from the derivations on the octonions. Its dimension is 248, and it has rank 8 (*i.e.*, there are 8 generators of the Cartan subalgebra). The root vectors are given by the 112 (eight dimensional) root vectors of $SO(16)$, which is contained as a subalgebra

$$(... \pm 1, ..., \pm 1, ...) \text{ all other entries } 0,$$

and by the following 128 (eight dimensional) vectors

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, ..., \pm\frac{1}{2}\right) \text{ even number of } - \text{ signs.}$$

It has the following Cartan matrix, which correspond to the metric of the

[1] I use capital letters for both the algebras and the groups. The only place in which E_8 is treated as a group here, is in the expression "gauge group".

root lattice

$$g_{ij}^{E_8} = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \\ . & & & & & & & \end{pmatrix} \tag{67}$$

Since E_8 has only one conjugacy class, namely the (0) (trivial) one, its root lattice is equal to the weight lattice, which implies that it is self-dual. By the form of its vectors, it is clear that it is also even.

The weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$ The other relevant lattice for the heterotic string is the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$. This group is the double cover of $SO(32)$, and has therefore the same dimension. Consider the Lie algebra of $SO(32)$, denoted D_{16}. As any D_n, it falls into four conjugacy classes, denoted by (0), (V) (Vector), (S) (Spinor), (C) (Conjugate spinor). The lattice with (0) and either (S) or (C) conjugacy class is the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$, which therefore contains the root lattice of $SO(32)$. It is generated by the following vectors:

- (0) Clearly, the vectors of the root lattice
- (S) $\mathbf{m} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, ... \pm \frac{1}{2},\right)$, with an even number of - signs, or
- (C) $\mathbf{m} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, ... \pm \frac{1}{2},\right)$, with an odd number of - signs.

To see that it is self-dual, note that (taking (S), with (C) the situation is the same since the resulting lattices are isomorphic) the weight lattice of $SO(32)$ is

$$\Lambda_w = \Lambda_R + 0_{(0)} \oplus (\Lambda_R + \mathbf{m}_{(S)}) \oplus (\Lambda_R + \mathbf{m}_{(V)}) \oplus (\Lambda_R + \mathbf{m}_{(C)}). \tag{68}$$

The Lie-algebra lattice is then

$$\Lambda_{\Gamma_{D_{16}}} = \Lambda_R + 0_{(0)} \oplus (\Lambda_R + \mathbf{m}_{(S)}). \tag{69}$$

Now, using the above exposed relations for the root and weight lattices one obtaines

$$\mathrm{vol}(\Lambda_w) = \frac{1}{4}\mathrm{vol}(\Lambda_R),\ \mathrm{vol}(\Lambda_w) = \mathrm{vol}(\Lambda_R)^{-1}. \quad (70)$$

This implies

$$\mathrm{vol}(\Lambda_w) = \frac{1}{2}, \quad \mathrm{vol}(\Lambda_R) = 2. \quad (71)$$

On the other hand,

$$\mathrm{vol}(\Gamma_{D_{16}}) = \frac{1}{2}\mathrm{vol}(\Lambda_R) = 1 \quad (72)$$

or in other words, the Lie-algebra lattice is unimodular. Being integer, it is also self-dual.

3.5 On-shell (massless) spectrum of the heterotic string

It is now easy to investigate the spectrum of the heterotic string. Following [2] relatively closely, for simplicity only the massless spectrum will be presented here. It is not much harder to work out the next levels, but the massless one already shows most of the good features of the heterotic string.

Since the heterotic string's Fock space corresponds to the tensor product of the Fock spaces of the two sectors, the spectrum is easily built by taking the tensor product of the states coming from the (once again: independent) left and right excitations. Not all combinations, however, are allowed. The spectrum is in fact subjected to the level-matching constraint, which will be discussed in the following.

Left moving excitations As usual, left moving excitations contain the tachyonic vacuum of the bosonic string, $|0\rangle$. At the massless level, one can excite the vacuum either with compactified or uncompactified oscillators: $\bar{\alpha}^{\mu}_{-1}|0\rangle$, or $\bar{\alpha}^{I}_{-1}|0\rangle$. The former transform like space-time vectors, whereas the latter correspond to the left-moving part of the abelian $U(1)^{16}$ gauge bosons that build the Cartan subalgebra of $E_8 \times E_8$ or $SO(32)$. There are, moreover, states in the soliton sector with non-trivial internal momenta $\mathbf{p}_L$. States $|\mathbf{p}_L\rangle = 2,\ N_L = 0$ are massless, $\mathbf{p}_L$ is a root vector of $E_8 \times E_8$ or $SO(32)$ and generate the non-Abelian gauge bosons of these groups.

Right moving excitations Right-moving excitations are simply those of the 10-dimensional superstring, and therefore the spectrum will be space-time supersymmetric. The NS tachyon $|0\rangle_{NS}$ is projected out ny GSO projection enforced by modular invariance. Lowest states are therefore the vector $b^{\mu}_{-1/2}$and the spinor $|S^{\alpha}\rangle$.

Left-Right level matching constraint Recall that a physical state has to satisfy

$$(L_0 - \bar{L}_0)\,|\text{phys}\rangle = 0 \tag{73}$$

This is because the unitary operator $U_\delta \equiv e^{2\pi i \frac{\delta}{l}(L_0 - \bar{L}_0}$ generates rigid sigma translations, *i.e.* $U_\delta^\dagger X^\mu(\sigma,\tau) U_\delta = X^\mu(\sigma+\delta,\tau)$, and no point on a closed string should be distinct. In the present case, this turns out to be equivalent to

$$m_L^2 = m_R^2 \leftrightarrow N_L + \frac{1}{2}\mathbf{p}_L^2 - 1 = \begin{cases} N_R & \text{R sector} \\ N_R - \frac{1}{2} & \text{NS sector} \end{cases} . \tag{74}$$

The only new term with respect to the superstring is $\mathbf{p}_L^2$, coming from the compactified momenta.

Massless spectrum of the heterotic string Putting the contents of the previous paragraph together, one can finally obtain the massless spectrum of the heterotic string. This is done, as already mentioned, by taking the tensor product of the right- and left moving sectors subjected to the level-matching constraint. Due to the right-moving supersymmetry, it is $\mathcal{N} = 1$ supersymmetric in 10 dimensions. The massless states are of four kinds:

1. $\bar{\alpha}_{-1}^{\mu}\,|0\rangle \otimes b_{-\frac{1}{2}}^{\nu}\,|0\rangle_{NS}$

   The components of the ten-dimensional graviton, antisymmetric tensor and dilaton;

2. $\bar{\alpha}_{-1}^{\mu}\,|0\rangle \otimes |S^\alpha\rangle_R$

   Their supersymmetric partners, gaugino and dilatino;

3. $\bar{\alpha}_{-1}^{I}\,|0\rangle \otimes b_{-\frac{1}{2}}^{\nu}\,|0\rangle_{NS}$ and $|\mathrm{p}_L^2 = 2\rangle \otimes b_{-\frac{1}{2}}^{\nu}\,|0\rangle_{NS}$

   The gauge bosons of $E_8 \times E_8$ or $SO(32)$, where the former corresponds to the gauge bosons of the Cartan subalgebra and the latter of the root vectors;

4. $\bar{\alpha}_{-1}^{I}\,|0\rangle \otimes |S^\alpha\rangle_R$ and $|\mathrm{p}_L^2 = 2\rangle \otimes |S^\alpha\rangle_R$

   Their supersymmetric partners (496), the so called gaugini.

One should note the absence of the tachyon, which has been projected out by means of the level-matching condition, the presence of the graviton, and the appearance of the internal gauge bosons. This summarizes the features of the heterotic string theory.

Relation between the $E_8 \times E_8$ and $SO(32)$ theories There is one final remark which is worthy to make. One may ask how different are the two theories (the one based on $E_8 \times E_8$, and the other on $SO(32)$) presented in this report. To the end of answering this question, there is one thing that can be computed easily. Recall that the partition functions of the compactified bosons encode the number of the solitonic states. Hence, in order to compare the number of states at each mass level it suffices to evaluate them. The calculation is done in [2] and it turns out that under this point of view the two theories are equivalent. However, the states are of course distributed in different ways.

4 Summary

The heterotic string introduces gauge symmetries in string theory surprisingly by mixing the right-moving sector of the bosonic string with the left-moving sector of the superstring. The exceeding bosonic dimensions are compactified on a 16 dimensional torus. Bosons on the torus generate internal shifts and thus local gauge symmetries.

Basing on modular invariance of the one-loop string partition function, two distinct shapes of the torus can be singled out, and correspondingly two different possible gauge groups of the theory: $E_8 \times E_8$ and $SO(32)$, which are much bigger than the gauge group of the Standard Model.

The heterotic string served as a starting point for the attempt to reproduce the Standard Model. Nevertheless, the heterotic string does not reproduce it correctly so far. Unfortunately, many other attempts to compactify on other manifolds and orbifolds have not lead to the desired result either

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