The Heterotic String

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Outline of this presentation

1. Introduction
2. Modular Invariance
3. Construction of the Heterotic String
4. Lie theory and lattices
5. Spectrum
Introduction

- Use many topics of previous talks to construct the heterotic string
  - Lie theory, modular invariance, superstrings, compactification...

- Heterotic string has been believed to be a starting point for reproducing the standard model
  - No tachyon, graviton, gauge symmetry (⇒ interactions)
Part I: Modular Invariance
String perturbation expansion

\[ A_n = \sum_{g=0}^{\infty} A^{(g)}_n \]

\[ = \sum_{g=0}^{\infty} C_{\Sigma_g} \int \mathcal{D}h \mathcal{D}X^\mu \int d^2z_1 \ldots d^2z_n V_1(z_1, \bar{z}_1) \ldots V_n(z_n, \bar{z}_n) e^{-S[X,h]} \]
Introduction Modular Invariance Construction of the Heterotic String Lie theory and lattices Spectrum

String perturbation expansion

\[ A_n = \sum_{g=0}^{\infty} A_n^{(g)} \]

\[ = \sum_{g=0}^{\infty} C_{\Sigma g} \int \mathcal{D} h \mathcal{D} X^{\mu} \int d^2 z_1 ... d^2 z_n V_1(z_1, \bar{z}_1) ... V_n(z_n, \bar{z}_n) e^{-S[X, h]} \]

From now on: concentrate on the one-loop vacuum amplitude

\[ A_0^{(1)} \sim \int_{\text{Torus}} \mathcal{D} h \mathcal{D} X^{\mu} e^{-S[X, h]} \]
Recall: Polyakov action is invariant under Weyl rescalings and diffeomorphisms of the world-sheet

\[ \begin{align*}
\text{Diffeo.} & \quad \delta X^\mu = \xi^\alpha \partial_\alpha X^\mu, \\
& \quad \delta h_{\alpha\beta} = -(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) \\
\text{Weyl} & \quad \delta X^\mu = 0 \\
& \quad \delta h_{\alpha\beta} = 2\Lambda h_{\alpha\beta}
\end{align*} \]

\( \Rightarrow \) Because of overcounting, path integral is highly divergent!
Redundancy and Modular Invariance

Try to compensate the overcounting

\[ \int \frac{Dh}{\text{Vol}(\text{Diff})\text{Vol}(\text{Weyl})} \]

⇒ Integration should be performed on a moduli space of metrics

\[ \mathcal{M}_g = \frac{\{\text{metrics}\}}{\{\text{Weyl}\} \times \{\text{diffeomorphisms}\}} \]

and the one-loop partition function must be modular invariant.
Tangent space decomposition

How to perform this in practice?

- Base space: modular parameters.
- Tangent space: Weyl and Diffeo.

\[
\delta h_{\alpha\beta} = \delta \Lambda h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + \sum_i \delta \tau_i \frac{\partial}{\partial \tau_i} h_{\alpha\beta}
\]

Weyl Diffeo. Moduli parameters

Define operator \( P \) (later purpose):

\[
(P\xi)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - (\nabla_\gamma \xi_\gamma) h_{\alpha\beta}
\]

Restrict integration to the slice above. Torus: next slide
**Moduli space of the torus**

Restriction to the slice of modular parameters for one-loop vacuum amplitude $A_0^{(1)}$. World-sheet is a torus. Recall: one modular parameter $\tau$

$$\mathcal{M}_1 = \mathbb{H}/\{\text{action of } \text{PSL}_2(\mathbb{Z})\}$$

How to obtain that?

- Teichmueller space (conformally inequivalent tori)
- Generators of the modular group (global diffeomorphisms)

$$T : \tau \rightarrow \tau + 1$$

$$S : \tau \rightarrow -\frac{1}{\tau}$$
The Fundamental Region

Result: inequivalent metrics reside on the fundamental region $\mathcal{F}$

$$\mathcal{F} = \{ z \in \mathbb{H} | |\text{Re}(z)|^2 \leq \frac{1}{2}, \ |z| \geq 1 \}$$
Orthogonal decomposition

Now, how to divide the measure in a suitable way?

- Need a notion of orthogonality:

\[(\delta h^{(1)}, \delta h^{(2)}) = \int \sqrt{h^{\alpha\gamma} h^{\beta\delta}} \delta h^{(1)}_{\alpha\beta} \delta h^{(2)}_{\gamma\delta}\]

- Decompose metric orthogonally into Weyl+Diffeo+Moduli. Yields Jacobian \(\mathcal{J}\)

\[\mathcal{D} h = \mathcal{J} \mathcal{D}\{\text{Weyl}\} \mathcal{D}\{\text{Diffeo}\} d\tau\]

- Would like to cancel out Weyl+Diffeo by further restricting integration on the fundamental region
Conformal Killing Group

Problem: Diffeo. and Weyl overlap. Restricting the integration does not completely eliminate the overcounting.

- Recall operator $P$:

$$ (P\xi)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - (\nabla_\gamma \xi^\gamma) h_{\alpha\beta} $$

  \begin{align*}
  \text{Diffeo.} & \\
  \text{Weyl.} & 
  \end{align*}

- Zero modes: ”Conformal Killing Vectors”. Form the ”Conformal Killing Group”.

⇒ have to divide integration measure by the ”volume” of the CKG.
Conformal Killing Group of the Torus

- Conformal Killing Group of the Torus (CKG): $U(1) \times U(1)$. 
- Generators: vector fields $\partial_z$ and $\partial_{\bar{z}}$. 
- Volume: 

$$\text{Vol}(\text{CKG}) \sim \text{Im}(\tau)$$

- Nice remark: by the Riemann-Roch theorem, 

$$\dim_C(\text{CKG}) = \{\text{number of moduli parameters}\}$$
Finally: the integration measure

From Vol(CKG) and zero-mode Jacobians, one obtains

\[ \int \frac{d^2\tau}{(\text{Im}\tau)^3} \]

However, measure by itself not modular invariant. Let \( \tau \rightarrow \frac{a\tau + b}{c\tau + d} \) be an arbitrary modular transformation. Then:

\[ d^2\tau \rightarrow |c\tau + d|^{-4} d^2\tau \]
\[ \text{Im}(\tau) \rightarrow |c\tau + d|^{-2}\text{Im}(\tau) \]

\[ \Rightarrow \text{Only } \frac{d^2\tau}{(\text{Im}\tau)^2} \text{ is modular invariant.} \]
Recall \( A_0^{(1)} \sim \int_{\text{Torus}} \mathcal{D} h \mathcal{D} X^\mu e^{-S[X,h]} \). We have decomposed the integration on the metrics. What about the rest?

- Recall that partition function of CFT’s on a Torus corresponds to the generating functional of a QFT with time compactified on a circle of radius \( R = \frac{1}{T} \) (temperature).

- Here: similar situation.

Nevertheless, should take some care with zero-modes and contributions

\[
(i.e. \int \frac{d^D p}{(2\pi)^D} \langle p | e^{-\pi \alpha \text{Im}(\tau)p^2} | p \rangle \sim \frac{1}{\sqrt{\text{Im}(\tau)}})
\]

and non-zero modes of Jacobian
Let $Z^*(\tau, \bar{\tau})$ denote the usual CFT partition function counting contributions in light-cone gauge but without those of zero-modes. Then:

$$A_0^{(1)} \sim \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im} \tau)^2} \frac{\text{Im}(\tau)^{-\frac{D}{2}+1} Z^*(\tau, \bar{\tau})}{Z}$$

where $D$ is the number of non-compact dimensions.

For any string theory, $Z(\tau, \bar{\tau})$ as defined above must be modular invariant.
Part II: Construction of the Heterotic String
Motivation

Recall: would like to reproduce the Standard Model.
Starting point:
  - Superstrings have no tachyons, contain bosons and fermions
  - Would like another feature: interactions. Try to implement gauge symmetries.
Note/recall: left- and right-moving sectors of the string are independent. E.g.

\[ [\bar{\alpha}_m^\mu, \alpha_n^\nu] = 0 \]

Idea: try to combine bosonic string and superstring.
Basic idea

- Take bosonic right-moving sector and supersymmetric left moving sector
- Recall: anomalies cancel in different space-time dimensions (26 and 10)
- Match dimensions by compactification!
  → Compactify 16 bosonic dimensions on a torus

Result:
- Result: String theory in 10D with gauge symmetry
1. **Left-moving coordinates**
   - 10 uncompactified bosonic fields $X_{\mu}^L(\tau + \sigma)$, ($\mu = 0, \ldots, 9$)
   - 16 internal bosons $X_{L}^I(\tau + \sigma)$ ($I = 1, \ldots, 16$) living on a torus

2. **Right-moving coordinates**
   - 10 uncompactified bosons $X_{\mu}^R(\tau - \sigma)$, ($\mu = 0, \ldots, 9$) with their
   - fermionic superpartners $\Psi_{\mu}^R(\tau - \sigma)$

How do the compactified coordinates look like? Consider compactified space (next slide).
Internal coordinates: discretized momenta

Recall one coordinate: single valuedness of the wave function
\[ \exp(i x p) \Rightarrow \text{discretized momenta} \]

\[ X^{25} \sim X^{25} + 2\pi R L, \quad L \in \mathbb{R} \]

\[ p^{25} = \frac{M}{R}, \quad M \in \mathbb{Z} \]

Here: 16 coordinates

\[ X^I \sim X^I + 2\pi \sum_{i=1}^{D} n^i e_i = X^I + 2\pi L^I, \quad n^i \in \mathbb{Z} \]

where the \( \{e_i\}_{i=1}^{D} \) are basis vectors of a lattice \( \Lambda \).

Momenta of additional bosons must be vectors of its 16-dimensional dual lattice \( \Gamma_{16} = \Lambda^* \).
**Basic definitions of lattices**

**Definition: Lattice**

A *n-dimensional lattice* \( \Gamma_n \) is a set of points in \( \mathbb{R}^n \) which can be written as integer combination of a set of basis vectors

\[
\Gamma_n = \{ x = \sum x^i e_i | x^i \in \mathbb{Z} \}
\]

**Definition: Dual lattice**

The *dual lattice* \( \Gamma_n^* \) is the lattice defined as

\[
\Gamma_n^* = \{ y | (y, x) \in \mathbb{Z}, \ x \in \Gamma_n \}
\]

**Definition: Even lattice**

A lattice is called *even* if for any two vectors \( x, y \in \Gamma \), \( (x, y) \in 2\mathbb{Z} \).
one-loop partition function

Recall Virasoro characters for bosons and fermions:

\[
\chi_{8-\text{fermions}}(\tau) = \frac{1}{2} \frac{1}{|\eta(\tau)|^4} \left( \theta(\tau)^4_3 - \theta(\tau)^4_4 - \theta(\tau)^4_2 \right)
\]

\[
\chi_{n-\text{bosons}}(\tau) = \left( \frac{1}{|\eta(\tau)|} \right)^n
\]

In addition, compactified bosons \((q = e^{2\pi i \tau})\):

\[
\chi_{16-\text{comp.bosons}}(\tau) = \frac{1}{|\eta(\tau)|^{16}} \sum_{p_L \in \Gamma_{16}} q^{\frac{1}{2} p_L^2}^2
\]
one-loop partition function

\[
Z^*_\text{het}(\tau, \bar{\tau}) = \chi_{8\text{-fermions}}(\tau)\chi_{8\text{-bosons}}(\tau)\chi_{8\text{-bosons}}(\bar{\tau})\chi_{16\text{-comp.bosons}}(\bar{\tau})
\]

\[
= \frac{1}{|\eta(\tau)|^4} \left( \theta(\tau)^4_3 - \theta(\tau)^4_4 - \theta(\tau)^4_2 \right) \left( \frac{1}{|\eta(\tau)|} \right)^8 \left( \frac{1}{|\eta(\bar{\tau})|} \right)^8
\]

\[
\times \frac{1}{|\eta(\bar{\tau})|^{16}} \sum_{p_L \in \Gamma_{16}} \bar{q}^2 \frac{1}{2} p_L^2
\]

\[
= \left( \frac{1}{[\eta(\bar{\tau})]^{24}} \sum_{p_L \in \Gamma_{16}} \bar{q}^2 \frac{1}{2} p_L^2 \right) \left( \frac{1}{[\eta(\tau)]^{12}} \left( \theta^4_3(\tau) - \theta^4_4(\tau) - \theta^4_2(\tau) \right) \right)
\]
By the previous discussion, the full partition function (with zero modes!)

\[
Z_{\text{het}}(\tau, \bar{\tau}) = \frac{1}{(\text{Im}\tau)^4} \left( \frac{1}{[\eta(\bar{\tau})]^{24}} \sum_{p_L \in \Gamma_{16}} \bar{q}^{\frac{1}{2}} p_L^2 \right) \times \\
\times \left( \frac{1}{[\eta(\tau)]^{12}} \left( \theta_3^4(\tau) - \theta_4^4(\tau) - \theta_2^4(\tau) \right) \right)
\]

has to be modular invariant.
Modular Invariance

Recall: generators of the modular group

\[ T : \tau \rightarrow \tau + 1 \]
\[ S : \tau \rightarrow -\frac{1}{\tau} \]

We know

<table>
<thead>
<tr>
<th>( T : \tau \rightarrow \tau + 1 )</th>
<th>( S : \tau \rightarrow -\frac{1}{\tau} )</th>
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<tr>
<td>( \eta(\tau) )</td>
<td>( e^{\frac{i\pi}{12}} \eta(\tau) )</td>
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<td>( \theta_2(\tau) )</td>
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Modular Invariance

- Only term of $Z_{het}$ whose transformation we don't know is the "soliton sum"

$$P(\tau) \equiv \sum_{p_L \in \Gamma_{16}} q^{\frac{1}{2}p_L^2}$$

- Requiring modular invariance of $Z_{het}$ leads to

$$P(\tau + 1) = P(\tau)$$

$$P\left(-\frac{1}{\tau}\right) = \tau^8 P(\tau)$$

- Translates into constraints on the allowed lattices!
The lattice $\Gamma_{16}$

Claim

$\Gamma_{16}$ must be an even, self-dual lattice
The lattice $\Gamma_{16}$

**Theorem**

In 16 dimensions, the only even, self-dual lattices are the direct product lattice $\Gamma_{E_8} \times \Gamma_{E_8}$, where $\Gamma_{E_8}$ is the root lattice of $E_8$, and $\Gamma_{D_{16}}$, the Lie algebra lattice of $SO(32)$ with the (0) and (S) conjugacy classes (or weight lattice of $\text{Spin}(32)/\mathbb{Z}_2$).
Part III: Lie Theory and Lattices
Let $g$ be a Lie-algebra. Recall:

- Cartan subalgebra: set of commuting generators $H^I$
- Diagonalize remaining generators $E^\alpha$ with respect to its elements

$$[H^I, E^\alpha] = \alpha^I E^\alpha$$

- Vectors $\alpha^I$ are called roots

Arbitrary integer linear combinations of roots $\Rightarrow$ root lattice $\Lambda_R$. 
Weight lattice $\Lambda_w$

- Take a particular representation of a Lie Group $G$. States can be denoted by

$$|m_l, D\rangle \quad l \in \{1...D\}$$

$D$: the dimension of the representation.

- Eigenstates of the Cartan subalgebra generators

$$H^l |c, D\rangle = m^l_l |m_l, D\rangle$$

$m_l$ are eigenvalues of the $H^l$: weight vectors.

Arbitrary integer linear combinations of weight vectors $\Rightarrow$ weight lattice $\Lambda_w$. 
The Lie-algebra lattice

Observation:

- $\Lambda_R \subset \Lambda_w$
- $\Lambda_R = \Lambda_w^*$
- $\text{vol}(\Lambda) = \text{vol}(\Lambda^*)^{-1}$

Overall note that

- $\Lambda_w = \Lambda_R \oplus (\Lambda_R + m_2) \oplus \ldots \oplus (\Lambda_R + m_{N_c})$, where $\{m_i\}_{i=1\ldots N_c}$ are representatives of conjugacy classes

Take only a subset of the conjugacy classes, closed under addition of all lattice vectors $\Rightarrow$ Lie algebra lattice
$E_8$ and $\Gamma_{E_8}$

What is $E_8$?

- An exceptional simple simply-laced Lie algebra
- Has dimension 248, rank 8
- Has only one conjugacy class $\Rightarrow \Gamma_{E_8}$ self-dual

What are its root vectors?

- 112 (8 dimensional) root vectors of $D_8$

  \[ (... \pm 1, ..., \pm 1, ...) \text{ all other entries 0} \]

- following 128 (8 dimensional) vectors

  \[ \left( \pm \frac{1}{2}, \pm \frac{1}{2}, ..., \pm \frac{1}{2} \right) \text{ even number of ” − ” signs} \]
Spin(32/\mathbb{Z}_2) and Γ_{D_{16}}

- Spin(32)/\mathbb{Z}_2 is the double cover of SO(32) ⇒ they have same dimension
- Recall that Lie algebra of SO(32), denoted D_{16}, has four conjugacy classes: trivial (0), Vector (V), Spinor (S), Conjugate spinor (C)
- Weight vectors of \text{Spin}(32)/\mathbb{Z}_2:
  - (0) ⇔ root lattice of SO(32)
    \[(k_1...k_n), \ k_i \in \mathbb{Z}, \sum_{i=1}^{n} k_i = \text{even}\]
  - (S): \textbf{m} = (±\frac{1}{2}, ±\frac{1}{2}, ... ± \frac{1}{2}), with an even number of "-" signs
Spin\((32/\mathbb{Z}_2)\) and \(\Gamma_{D_{16}}\)

How to see that \(\Gamma_{D_{16}}\), Lie algebra lattice of \(SO(32)\), is self-dual? Consider weight lattice \(\Lambda_w\) and root lattice \(\Lambda_R\) of \(SO(32)\)

- \(\Lambda_w = \Lambda_R + 0(0) \oplus (\Lambda_R + m_S) \oplus (\Lambda_R + m_V) \oplus (\Lambda_R + m_C)\)
- \(\Lambda_{\Gamma_{D_{16}}} = \Lambda_R + 0(0) \oplus (\Lambda_R + m_S)\)
- \(\text{vol}(\Lambda_w) = \frac{1}{4} \text{vol}(\Lambda_R), \quad \text{vol}(\Lambda_w) = \text{vol}(\Lambda_R)^{-1}\)
- \(\Rightarrow \text{vol}(\Lambda_w) = \frac{1}{2}, \quad \text{vol}(\Lambda_R) = 2\)
- \(\text{vol}(\Gamma_{D_{16}}) = \frac{1}{2} \text{vol}(\Lambda_R) = 1 \Rightarrow \text{unimodular}\)
- Consider vectors of \((S)\): it is integer
- self-dual \(\Leftrightarrow\) unimodular and integer
Part IV: Spectrum
Spectrum and the level matching condition

- Spectrum is constructed by taking the tensor product of right- and left-moving excitations
- Right-moving sector: $\mathcal{N} = 1$ supersymmetric in 10 dimensions
- Level matching condition

\[
m_L^2 = m_R^2 \iff N_L + \frac{1}{2} p_L^2 - 1 = \begin{cases} N_R & \text{R sector} \\ N_R - \frac{1}{2} & \text{NS sector} \end{cases}
\]
The Massless States

1. Components of graviton, antisymmetric tensor and dilaton (NS-sector):
   \[ \bar{\alpha}^\mu_{-1} |0\rangle \otimes b^{\nu}_{-\frac{1}{2}} |0\rangle_{NS} \]

2. Supersymmetric partners gaugino, dilatino (R-sector):
   \[ \bar{\alpha}^\mu_{-1} |0\rangle \otimes |S^\alpha\rangle_R \]

3. The gauge bosons of \( E_8 \times E_8 \) or \( SO(32) \)
   - \[ \bar{\alpha}^I_{-1} |0\rangle \otimes |S^\alpha\rangle_R \] Gauge bosons of the Cartan subalgebra
   - \[ |p^2_L = 2\rangle \otimes |S^\alpha\rangle_R \] Root vectors

4. 496 supersymmetric partners, gaugini:
   \[ \bar{\alpha}^I_{-1} |0\rangle \otimes |S^\alpha\rangle_R \text{ and } |p^2_L = 2\rangle \otimes |S^\alpha\rangle_R \]
Conclusions

- Singled out two distinct compactifications (on a torus) for the heterotic string from the modular invariance of the one-loop vacuum amplitude
- Many attempts to compactify on other manifolds/orbifolds
- Does any other heterotic String Theory reproduce the Standard Model? Unfortunately, not yet completely
Books:


Article:

Thank you!