Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Report for
Proseminar in Theoretical Physics

Superstrings

Author:
Imre Majer

Supervisor:
Cristian Vergu

Abstract

During the course of the previous chapters of the proseminar, we introduced and quantized both the bosonic and fermionic strings and determined that the theory only containing bosonic strings requires $D = 26$ dimensions, whereas if it contains fermionic strings as well, the critical dimension is $D = 10$. We have also seen that the latter is formulated such that it incorporates manifest worldsheet supersymmetry (Neveu–Schwarz–Ramond formulation). By naïvely building up the spectrum of states using the tools previously introduced, we will see that this theory in itself is inconsistent and contains several unwanted features. To fix these, a procedure called the GSO projection is needed. We will first define and use it to obtain a spacetime supersymmetric spectrum which is free from the mentioned inconsistencies. Then, we will justify the seemingly arbitrary GSO conditions by requiring that the theory be modular invariant. In the final part of this report, we take a completely different approach, and formulate another theory with manifest spacetime supersymmetry (Green–Schwarz fomulation). Spacetime supersymmetry is an elegant property, and so, many constraints and restrictions are imposed in both formulations merely for the sake of a supersymmetric result. Without rigorous proof, we will show that with these constraints, the two formulations are actually equivalent.

Contents

1 Neveu–Schwarz–Ramond Formulation

1.1 Quick Revision

The Neveu–Schwarz–Ramond (NSR) formulation is what we have learned so far, so let us just briefly recall the tools that we will need for creating and analysing the spectrum. Recall the superstring action

$$S_{\text{NSR}} = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu - i\bar{\psi}_\mu \rho^\alpha \partial_\alpha \psi^\mu\right). \tag{1.1}$$

Both X^μ and ψ^μ are spacetime vectors, meaning that they transform under the vector representation of the SO(1,$D-1$) Lorentz group, but while X^μ is just a scalar on the worldsheet, ψ^μ is a Majorana 2-spinor (and thus carries two spinor indices). ρ^α are the two dimensional Dirac matrices, and, as they can be chosen to be purely imaginary, the Majorana representation (*i.e.* reality) for ψ^μ is indeed possible.

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.2}$$

This action is invariant under infinitesimal worldsheet supersymmetry transformations of the form

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon}\psi^\mu, \\ \delta\psi^\mu &= -i\rho^\alpha \partial_\alpha X^\mu \epsilon. \end{aligned} \tag{1.3}$$

Here, ϵ is an infinitesimal worldsheet Majorana 2-spinor.
We have seen that quantization in the light cone gauge is one option, and the procedure only keeps the global Lorentz symmetry if we work in $D = 10$ spacetime dimensions. The light cone gauge choice fixes the $\mu = 0, 9$ components, and for the remaining eight transverse ones, we use the Latin index $i = 1, \ldots, 8$. The action in the light cone gauge is thus

$$S^{l.c.}_{\text{NSR}} = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_\alpha X^i \partial^\alpha X^i - i\bar{\psi}^i \rho^\alpha \partial_\alpha \psi^i\right). \tag{1.4}$$

This still has a global SO(8) rotational symmetry, so both X^i and ψ^i are vector representations of SO(8), for which we use the symbol $\mathbf{8_v}$. As a side note, let us make an early mention that SO(8) has two other fundamental representations: the spinor ($\mathbf{8_s}$) and the conjugate spinor ($\mathbf{8_c}$).

A closed string is just the tensor product of its right- and left-moving parts, which are separately equivalent to an open string, up to a factor of two in their masses. For most of the time, we will consider a closed string and just work with the right-moving part, but, of course, everything is exactly the same for the left-moving part, and can be related to the open string.

As discussed previously, the boundary conditions on a closed string are strictly periodic for the bosonic part: $X^\mu(0,\tau) = X^\mu(\pi,\tau)$, leading to the oscillator expansion

$$X^\mu_R(\sigma) = \sum_{n\in\mathbb{Z}} \alpha_n e^{-2in(\sigma-\tau)}. \tag{1.5}$$

For the fermionic right-movers, we can impose both periodicity (R-sector) and antiperiodicity (NS-sector), leading to two different oscillator expansions.

$$\psi_-^\mu(\sigma) = \sum_{n\in\mathbb{Z}} d_n e^{-2in(\sigma-\tau)} \qquad \text{R-sector,} \tag{1.6}$$

$$\psi_-^\mu(\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r e^{-2ir(\sigma-\tau)} \qquad \text{NS-sector.} \tag{1.7}$$

The reality condition for both X^μ and ψ^μ implies that

$$(\alpha_n)^\dagger = \alpha_{-n}, \qquad (d_n)^\dagger = d_{-n}, \qquad (b_r)^\dagger = b_{-r}. \tag{1.8}$$

The oscillator ground state in both sectors is defined such that the positive-integer-valued oscillators annihilate it.

$$\alpha_n |0\rangle = d_n |0\rangle = b_r |0\rangle = 0 \qquad \forall n, r > 0 \tag{1.9}$$

Ergo, the positive-valued oscillators are annhilation operators and the negative-valued ones are creation operators. These operators also satisfy the canonical (anti)commutation relations.

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \qquad \{d_m^\mu, d_n^\nu\} = \delta_{m+n}\eta^{\mu\nu}, \qquad \{b_r^\mu, b_s^\nu\} = \delta_{r+s}\eta^{\mu\nu}. \tag{1.10}$$

The mass-squared operator basically just counts the number of oscillator excitations.

$$\alpha' m^2 = \underbrace{\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i}_{N^{(\alpha)}} + \underbrace{\sum_{m=1}^{\infty} m d_{-m}^i d_m^i}_{N^{(d)}} - \frac{1}{2} \qquad \text{NS-sector,} \tag{1.11}$$

$$\alpha' m^2 = \underbrace{\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i}_{N^{(\alpha)}} + \underbrace{\sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i}_{N^{(b)}} \qquad \text{R-sector.} \tag{1.12}$$

1.2 The Naïve Spectrum

The field operators X^μ and ψ^μ in the NSR formulation are both spacetime vectors, so, naturally, the oscillators must be spacetime vectors as well. No matter how many operators we apply to a state, the property of whether it is a fermionic or bosonic state cannot be changed. The entire spectrum of states is bosonic/fermionic if it is built on a bosonic/fermionic ground state, respectively. Therefore, it is important to investigate the ground state in both the NS- and the R-sector.

1.2.1 The NS-Sector Ground State

The NS-sector encompasses spacetime bosons. We will label the scalar ground state here simply by $|0\rangle$. Looking at (1.11), we see that, due to the anomaly term, even if there are no excitations, we get a negative mass-squared value, *i.e.* the ground state in the NS-sector is tachyonic.

1.2.2 The R-Sector Ground State

The ground state in this sector is massless, as there is no anomaly in (1.12). This massless ground state is degenerate, as the d_0^μ operators map one ground state to another with the same mass-squared eigenvalue[1]. The degeneracy is characterized by the fact that these d_0^μ operators obey the canonical anticommutation relation, which here is nothing but the SO(1,9) Clifford algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}. \tag{1.13}$$

If the operators that map between the ground states obey the SO(1,9) Clifford algebra, the various ground states can be gathered into a spinor representation of SO(1,9). Therefore, the ground state in the R-sector is a massless 10-dimensional spinor

$$|\chi\rangle = \chi^c \, |c\rangle \, , \qquad c = 1, \dots, 32. \tag{1.14}$$

A spinor in D dimensions has $2^{[D/2]}$ generally complex components, which now means $2^5 = 32$.

But the story of this ground state does not end here. As we mentioned in the abstract, a driving force in our arguments is often that we want the resulting spectrum to be spacetime supersymmetric. We can take a step toward this goal by imposing several constraints on this ground state. (Although the reason why this is necessary for spacetime supersymmetry will only become apparent later in Chapter 3.) The three conditions the ground state has to satisfy are the following:

- Majorana constraint,
- Weyl constraint,
- Dirac equation.

Each condition will reduce the initial 64 degrees of freedom (32 complex numbers) by a factor of two, so, in the end, we will end up with only 8. We will now cover each of these conditions step by step.

Majorana constraint: A simple reality condition for the spinor field χ. However, the fact that this constraint is possible is not trivial. The spinor field also has to satisfy the massless[2] Dirac equation[3]

$$i\Gamma^\mu \partial_\mu \chi = 0. \tag{1.15}$$

[1] This is not the case the NS-sector, where there are no zero-valued oscillators.

[2] The full Dirac equation is $(i\Gamma^\mu\partial_\mu - m)\chi = 0$, but the ground state is massless in the R-sector.

[3] Here, we do not cover how obeying the Dirac equation reduces the number of degrees of freedom, we just use the fact that the spinor field obeys it.

Γ^μ are generally ten complex 32-dimensional Dirac matrices obeying the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}. \tag{1.16}$$

If it is possible to find a representation of the Dirac matrices where all of them are purely real or imaginary ("Majorana representation"), then the reality condition is possible ("Majorana spinor")[4]. In the following, we will construct purely imaginary Dirac matrices. For this, let us investigate SO(8), the transverse subgroup of SO(1,9). This will not only help us construct our 32-dimensional Dirac matrices (always denoted by Γ^μ), but will come in handy later as well.

First, let us define eight 16-dimensionsal Dirac matrices, denoted by γ^i the following way:

$$\gamma^i = \begin{pmatrix} 0 & \gamma^i_{a\bar{a}} \\ \gamma^i_{\bar{a}a} & 0 \end{pmatrix}_{16\times 16}. \tag{1.17}$$

Where $\gamma^i_{a\bar{a}}$ (and its transposed $\gamma^i_{\bar{a}a}$) are **real**, symmetric matrices defined as

$$\begin{aligned}
\gamma^1_{a\bar{a}} &= i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2, & \gamma^2_{a\bar{a}} &= \mathbb{1} \otimes \sigma_1 \otimes i\sigma_2, \\
\gamma^3_{a\bar{a}} &= \mathbb{1} \otimes \sigma_3 \otimes i\sigma_2, & \gamma^4_{a\bar{a}} &= \sigma_1 \otimes i\sigma_2 \otimes \mathbb{1}, \\
\gamma^5_{a\bar{a}} &= \sigma_3 \otimes i\sigma_2 \otimes \mathbb{1}, & \gamma^6_{a\bar{a}} &= i\sigma_2 \otimes \mathbb{1} \otimes \sigma_1, \\
\gamma^7_{a\bar{a}} &= i\sigma_2 \otimes \mathbb{1} \otimes \sigma_3, & \gamma^8_{a\bar{a}} &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}.
\end{aligned} \tag{1.18}$$

σ^j are the regular Pauli matrices, and $\mathbb{1}$ is the 2-dimensional identity matrix. It is easy to check that these matrices obey the Clifford algebra

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij}. \tag{1.19}$$

Also, from these, one can construct the SO(8) transformation matrices for spinors[5]. It can be checked that a 16-dimensional spinor λ under this transformation transforms reducibly. The irreducible parts are called the spinor ($\mathbf{8_s}$) and the conjugate spinor ($\mathbf{8_c}$) representations of SO(8). In the following, we will always use the symbol $\lambda = (\lambda^a_s, \lambda^{\bar{a}}_c)$ for the reducible 16-spinor of $\mathbf{8_s} \oplus \mathbf{8_c}$. λ^a_s is the spinor, $\lambda^{\bar{a}}_c$ is the conjugate spinor part. Furthermore, we denote the spinor index with $a = 1, \ldots, 8$, the conjugate spinor index with $\bar{a} = 1, \ldots, 8$, and the vector index with $i = 1, \ldots, 8$.

With the help of these 16-dimensional γ^i matrices, we can construct the 32-dimensional Γ^μ matrices

$$\begin{aligned}
\Gamma^0 &= \sigma_2 \otimes \mathbb{1}_{16}, \\
\Gamma^i &= i\sigma_1 \otimes \gamma^i, \qquad i = 1, \ldots, 8, \\
\Gamma^9 &= i\sigma_3 \otimes \mathbb{1}_{16}.
\end{aligned} \tag{1.20}$$

[4]The SO(1,9) Lorentz generators for spinors are $\frac{i}{4}\left[\Gamma^\mu, \Gamma^\nu\right]$.

[5]The SO(8) generators for spinors are $\frac{i}{4}\left[\gamma^i, \gamma^j\right]$.

Again, it is easily checked that the Dirac matrices defined in this way obey the Clifford algebra (1.16). Due to the definition of (1.17) and (1.18), the γ^i matrices are all real, so by creating the Γ^μ matrices by means of (1.20), we obtain purely imaginary Dirac matrices. Thus, the Majorana condition is possible[6], χ is a real spinor, containing 32 real degrees of freedom.

Weyl constraint: The spinor field has a definite chirality ("Weyl spinor"). First, let us define the chirality operator Γ_{11} similar to γ_5 in four dimensions.

$$\Gamma_{11} = \Gamma^0\Gamma^1 \cdots \Gamma^9 = -\sigma_1 \otimes \begin{pmatrix} \mathbb{1}_8 & \\ & -\mathbb{1}_8 \end{pmatrix}. \tag{1.21}$$

This operator has only eigenvalues $+1$ and -1, and, for each eigenvalue, the same number of eigenvectors. Thus, requiring that the spinor should have definite chirality, *i.e.* satisfy

$$\Gamma_{11} |\chi\rangle = \pm |\chi\rangle , \tag{1.22}$$

will again eliminate half of the degrees of freedom[7] ($\mathbf{16_s} \oplus \mathbf{16_c}$), resulting in 16 ($\mathbf{8_s} \oplus \mathbf{8_c}$).

Dirac equation: The spinor field should be propagating, *i.e.* satisfy the Dirac equation, so let us repeat it again:

$$i\Gamma^\mu \partial_\mu \chi = 0. \tag{1.23}$$

For Weyl spinors $\lambda = (\lambda_s, \lambda_c)$, this reduces to (see Appendix A)

$$\begin{aligned} (\partial_0 \pm \partial_9)\, \lambda_s^a + \gamma^i_{a\bar{a}} \partial_i \lambda_c^{\bar{a}} = 0, \qquad & a = 1, \ldots, 8, \\ (\partial_0 \mp \partial_9)\, \lambda_c^{\bar{a}} + \gamma^i_{\bar{a}a} \partial_i \lambda_s^{a} = 0, \qquad & \bar{a} = 1, \ldots, 8, \end{aligned} \tag{1.24}$$

for chirality $\Gamma_{11} |\chi\rangle = \pm |\chi\rangle$. These equations clearly relate the spinor and the conjugate spinor representations; they are not independent anymore, so the number of independent components reduces again, reaching 8. Conventionally, for positive chirality, we say that the eight degrees of freedom form an $\mathbf{8_s}$ spinor λ_s, and, for negative chirality, an $\mathbf{8_c}$ conjugate spinor λ_c.

Thus, the ground state in the R-sector forms an $\mathbf{8_s} \oplus \mathbf{8_c}$ multiplet.

[6] The possibility to impose the Majorana condition on spinors of SO(p,q) depends heavily on p and q. For SO(1,$D-1$), it is generally possible for $D = 2, 3, 4$ (mod 8) cases.

[7] Majorana and Weyl conditions for an SO(1,$D-1$) spinor are only compatible in $D = 2$ (mod 8) dimensions. For example, in four dimensions, it is not possible to impose both at the same time, as γ_5 is imaginary in the Majorana representation.

1.2.3 Building up the Spectrum

Now that the ground states have been clarified in both sectors, by applying creation operators α_{-n}, b_{-r} in the NS-sector, and α_{-n}, d_{-m} in the R-sector, the higher massive states can be reached. The first few can be seen in Table [1]. For a detailed explanation of the meaning of the representation contents, see Appendix B. The table also contains a middle column with the eigenvalues of the operators G and $\bar{\Gamma}$, about which we will soon talk.

1.3 The GSO Projection

So far, the NSR formulation looks nice, but it is actually inconsistent. This is still not apparent, but the spectrum conflicts with modular invariance. Besides this, it also has several unflattering features. First of all, the spectrum contains a tachyon, which is undesirable because it makes the vacuum unstable. Secondly, we are not used to anticommuting operators that map bosons to bosons, which is the case in the NS-sector. This is actually not in conflict with the spin statistics theorem, but certainly it is something very unusual and unnerving. And finally, as we emphasized before, we would like to obtain a spacetime supersymmetric result in the end, and one of the necessary conditions for that is to have the same number of bosons and fermions on each mass level. In the case of our spectrum, the problem is not only that, on the mass levels existing in both the bosonic (NS) and the fermionic (R) sectors, the degrees of freedom are not equal, but also that there are even mass levels in the bosonic sector that do not have a counterpart in the fermionic one. Spacetime supersymmetry certainly is not possible this way.

To fix these problematic features, F. Gliozzi, J. Scherk, and D. I. Olive suggested the truncation of the spectrum. After them, the procedure is called the GSO projection. Here we will first define the GSO conditions, and then see how it helps us acquire a spacetime supersymmetric spectrum free from the aforementioned unwanted properties. Deciding which part to discard from the spectrum will seem a bit arbitrary at first, but in Chapter 2 we will prove that the GSO projection is actually a consequence of modular invariance.

1.3.1 Truncation of the NS-Sector

Let $|\varphi_0\rangle$ be a bosonic state. Act with a number of creation operators on this state:

$$|\varphi\rangle = b^{i_1}_{-r_1} b^{i_2}_{-r_2} \cdots b^{i_n}_{-r_n} |\varphi_0\rangle \,. \tag{1.25}$$

Because all the operators in the NSR formulation are spacetime vectors, the state $|\varphi\rangle$ is a bosonic state as well. If n is even, then the product of the oscillators is a commuting operator, and there is nothing unusual in a commuting operator mapping between bosonic states. However, if n is odd, the product is anticommuting, and this is one of the problematic features of this model, as we really do not like anticommuting operators mapping between bosonic states. The GSO projection for the NS-sector is thus the elimination of the states $|\varphi\rangle$, which are created using odd number of creation oscillators. However, this is not well defined, as every state in the NS-sector is bosonic, we are still free to choose the states of reference $|\varphi_0\rangle$. We will thus choose these states such that the result also appeals to some other requirements, namely that the tachyonic state should

<table>
<tr><th>$\alpha' m^2$</th><th>states and their SO(8) representation contents</th><th>G (NS)
$\bar{\Gamma}$ (R)</th><th>little group</th><th>representation contents with respect to the little group</th></tr>
<tr><td colspan="5">NS-sector (bosons)</td></tr>
<tr><td>$-\frac{1}{2}$</td><td>$\vert 0\rangle$
$\mathbf{1}$</td><td>-1</td><td>SO(9)</td><td>$\mathbf{1}$</td></tr>
<tr><td>0</td><td>$b^i_{-1/2}\vert 0\rangle$
$\mathbf{8_v}$</td><td>$+1$</td><td>SO(8)</td><td>$\mathbf{8_v}$</td></tr>
<tr><td>$+\frac{1}{2}$</td><td>$\alpha^i_{-1}\vert 0\rangle$ $\quad$ $b^i_{-1/2}b^j_{-1/2}\vert 0\rangle$
$\mathbf{8_v}$ $\quad$ $\mathbf{28}$</td><td>-1</td><td>SO(9)</td><td>$\mathbf{36}$</td></tr>
<tr><td>$+1$</td><td>$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\vert 0\rangle$
$\mathbf{56_v}$
$\alpha^i_{-1}b^j_{-1/2}\vert 0\rangle$ $\quad$ $b^i_{-3/2}\vert 0\rangle$
$\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}$ $\quad$ $\mathbf{8_v}$</td><td>$+1$</td><td>SO(9)</td><td>$\mathbf{84}\oplus\mathbf{44}$</td></tr>
<tr><td colspan="5">R-sector (fermions)</td></tr>
<tr><td rowspan="2">0</td><td>$\vert a\rangle$
$\mathbf{8_s}$</td><td>$+1$</td><td rowspan="2">SO(8)</td><td>$\mathbf{8_s}$</td></tr>
<tr><td>$\vert \bar{a}\rangle$
$\mathbf{8_c}$</td><td>-1</td><td>$\mathbf{8_c}$</td></tr>
<tr><td rowspan="2">$+1$</td><td>$\alpha^i_{-1}\vert a\rangle$ $\quad$ $d^i_{-1}\vert \bar{a}\rangle$
$\mathbf{8_c}\oplus\mathbf{56_c}$ $\quad$ $\mathbf{8_s}\oplus\mathbf{56_s}$</td><td>$+1$</td><td rowspan="2">SO(9)</td><td>$\mathbf{128}$</td></tr>
<tr><td>$\alpha^i_{-1}\vert \bar{a}\rangle$ $\quad$ $d^i_{-1}\vert a\rangle$
$\mathbf{8_s}\oplus\mathbf{56_s}$ $\quad$ $\mathbf{8_c}\oplus\mathbf{56_c}$</td><td>$-1$</td><td>$\mathbf{128}$</td></tr>
</table>

Table 1: The spectrum of the first few right-moving (open string) states.

be eliminated, and also all the states of those massive levels which do not exist in the fermionic sector.

Formally, we define a quantum number G as

$$G = -(-1)^F, \qquad \text{where } F = \sum_{r=1/2}^{\infty} b^i_{-r} b^i_r, \tag{1.26}$$

and demand that for the states kept

$$G|\varphi\rangle = +|\varphi\rangle. \tag{1.27}$$

The states with eigenvalue -1 are discarded. This is the GSO projection for the bosonic (NS) sector.

1.3.2 Truncation of the R-Sector

After the NS-sector GSO projection, we still have not acquired the necessary conditions for a supersymmetric spectrum. On the fermionic mass levels, there are twice as many states as the bosonic levels. So the idea of the R-sector GSO projection is just to discard half of them. For this, we first generalize the chirality operator Γ_{11} to massive levels:

$$\bar{\Gamma} = \Gamma_{11}(-1)^F, \qquad \text{here } F = \sum_{m=1}^{\infty} d^i_{-m} d^i_m. \tag{1.28}$$

Similarly to the Weyl condition[8], we demand that

$$\bar{\Gamma}|\chi\rangle = \pm|\chi\rangle. \tag{1.29}$$

Therefore, in the R-sector, two different GSO projections are possible. Either we keep the spinor states with $+1$ eigenvalue for $\bar{\Gamma}$ or those with -1. This will certainly eliminate half the states due to $\{\bar{\Gamma}, d^\mu_n\} = 0$, and for the ground state: $\bar{\Gamma} = \Gamma_{11}$, so by anticommuting the operator past all the creation operators, the GSO condition of a state in the R-sector reduces to the Weyl condition of the ground state upon which it is built.

The middle column of Table [1] corresponds to the G (in the NS-sector) and $\bar{\Gamma}$ (in the R-sector) eigenvalues of the states. After applying the GSO projection, a necessary condition for a spacetime supersymmetric spectrum is guaranteed, as there are now equal number of bosonic and fermionic degrees of freedom on each of the mass levels. Also, we got rid of the tachyon, and now only commuting operators can map between two bosonic states. But so far, we have only investigated the right-moving spectrum of a closed string (or equivalently, the spectrum of an open string).

[8]This does not mean that the massive states are Weyl spinors, as massive states cannot have definite chirality. This is because while the chirality operator Γ_{11} anticommutes with Γ^μ, in the Dirac equation now a mass term also appears, which is just a commuting constant. Thus a massive spinor cannot have definite $+1$ or -1 eigenvalue for Γ_{11}, so it cannot have definite chirality.

1.4 Type II Theories: Closed String Spectrum

A closed string state is formed by taking the tensor product of a right-moving state with a left-moving one. The only restriction is that we can only tensor together states which belong to the same mass levels[9].

On a closed string, we thus have four different sectors: (NS,NS) and (R,R) for bosons, and (NS,R) and (R,NS) for fermions. We apply the GSO projection separately for the right- and the left-movers. While the GSO projection in the NS-sector for both movers is the same $G_L = G_R = +1$, we obtain two different theories by demanding opposite or same eigenvalues of the $\bar{\Gamma}$ operator for the right- and left-movers.

- Type IIA theory: $\bar{\Gamma}_L = -\bar{\Gamma}_R = 1$
- Type IIB theory: $\bar{\Gamma}_L = \bar{\Gamma}_R = 1$

The reason why these theories are called type II will be clarified later in Chapter 3. Table [2] shows the closed string spectrum before the GSO projection until the massless level.

Applying the GSO projection leads to the following massless spectrum for the **type IIA** theory:

$$\begin{array}{ll} \text{Bosons:} & [\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}] \,\oplus\, [\mathbf{8_v} \oplus \mathbf{56_v}] \\ \text{Fermions:} & [\mathbf{8_c} \oplus \mathbf{56_c}] \,\oplus\, [\mathbf{8_s} \oplus \mathbf{56_s}] \end{array} \tag{1.30}$$

In detail, the real scalar **1** corresponds to the dilaton, the **28** is a rank-2 antisymmetric tensor without a specific name, the $\mathbf{35_v}$ is a rank-2 symmetric traceless tensor called the graviton, and the $\mathbf{8_v}$ and $\mathbf{56_v}$ are a vector and a rank-3 antisymmetric tensor field, respectively, again without specific names. This concludes the massless bosonic degrees of freedom, which together number 128. The theory is supersymmetric: the massless fermionic sector contains the same number of degrees of freedom, which are two spin $^3/_2$ gravitinos of opposite handedness ($\mathbf{56_s}$ and $\mathbf{56_c}$), and two spin $^1/_2$ dilatinos, again one for each chirality ($\mathbf{8_s}$ and $\mathbf{8_c}$). Both chiralities are present for the fermions, there is no distinguished one, therefore the theory is called non-chiral.

The other possible GSO projection yields the following massless spectrum for the **type IIB** theory:

$$\begin{array}{ll} \text{Bosons:} & [\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}] \,\oplus\, [\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_s}] \\ \text{Fermions:} & [\mathbf{8_c} \oplus \mathbf{56_c}] \,\oplus\, [\mathbf{8_c} \oplus \mathbf{56_c}] \end{array} \tag{1.31}$$

The new $\mathbf{35_s}$ here is a rank-4 antisymmetric, self-dual tensor field, and in contrast to the type IIA theory, there are two real scalars and two antisymmetric rank-2 tensors. Furthermore, this theory is chiral, since both gravitinos and dilatinos have the same handedness. This theory is also supersymmetric.

The NS-sector GSO projection takes care of the tachyon, so none of the type II theories have it, and also, for the massive states, the two theories yield the same spectrum.

[9] The Virasoro generators have to be equal for the right- and the left-moving parts: $L_0 = \tilde{L}_0$, corresponding to the fact, that there are no distinct points on a closed string. This implies $m_L^2 = m_R^2$.

$\alpha' m^2$	states and their SO(8) representation contents	G_{L} (NS) $\bar{\Gamma}_{\mathrm{L}}$ (R)	G_{R} (NS) $\bar{\Gamma}_{\mathrm{R}}$ (R)	little group	representation contents with respect to the little group
(NS,NS)-sector (bosons)					
-2	$\lvert 0\rangle_{\mathrm{L}} \otimes \lvert 0\rangle_{\mathrm{R}}$ $\mathbf{1} \otimes \mathbf{1}$	-1	-1	SO(9)	$\mathbf{1}$
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_{\mathrm{L}} \otimes b^j_{-1/2} \lvert 0\rangle_{\mathrm{R}}$ $\mathbf{8_v} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$
(R,R)-sector (bosons)					
0	$\lvert a\rangle_{\mathrm{L}} \otimes \lvert b\rangle_{\mathrm{R}}$ $\mathbf{8_s} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_s}$
	$\lvert \bar{a}\rangle_{\mathrm{L}} \otimes \lvert \bar{b}\rangle_{\mathrm{R}}$ $\mathbf{8_c} \otimes \mathbf{8_c}$	-1	-1		$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c}$
	$\lvert \bar{a}\rangle_{\mathrm{L}} \otimes \lvert b\rangle_{\mathrm{R}}$ $\mathbf{8_c} \otimes \mathbf{8_s}$	-1	$+1$		$\mathbf{8_v} \oplus \mathbf{56_v}$
	$\lvert a\rangle_{\mathrm{L}} \otimes \lvert \bar{b}\rangle_{\mathrm{R}}$ $\mathbf{8_s} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_v} \oplus \mathbf{56_v}$
(R,NS)-sector (fermions)					
0	$\lvert a\rangle_{\mathrm{L}} \otimes b^i_{-1/2} \lvert 0\rangle_{\mathrm{R}}$ $\mathbf{8_s} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	$\lvert \bar{a}\rangle_{\mathrm{L}} \otimes b^i_{-1/2} \lvert 0\rangle_{\mathrm{R}}$ $\mathbf{8_c} \otimes \mathbf{8_v}$	-1	$+1$		$\mathbf{8_s} \oplus \mathbf{56_s}$
(NS,R)-sector (fermions)					
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_{\mathrm{L}} \otimes \lvert a\rangle_{\mathrm{R}}$ $\mathbf{8_v} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	$\tilde{b}^i_{-1/2} \lvert 0\rangle_{\mathrm{L}} \otimes \lvert \bar{a}\rangle_{\mathrm{R}}$ $\mathbf{8_v} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_s} \oplus \mathbf{56_s}$

Table 2: The closed string spectrum up to the massless level before applying the GSO projection, *i.e.* before specifying theory type IIA, or type IIB.

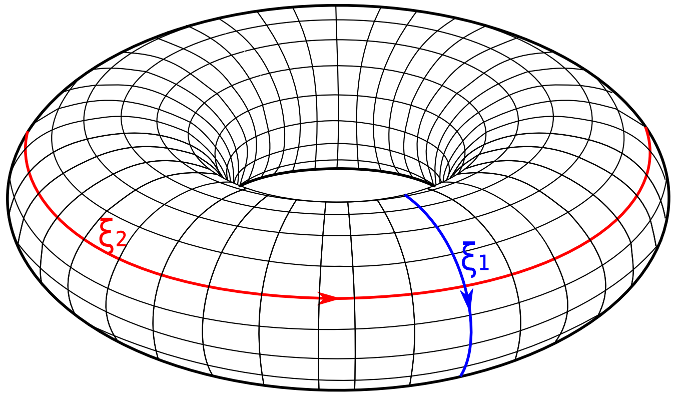

Figure 1: A simple torus parametrized by ξ^1 (space direction) and ξ^2 (Euclidean time direction).

2 Modular Invariance

This chapter is dedicated to justifying the seemingly arbitrary GSO conditions of the previous chapter, by requiring a basic geometrical symmetry: modular invariance. We will also see, that this will ensure that the vacuum energy get the same contribution from the bosons and fermions, so, all in all, it will vanish[10]. This is again a necessary condition for a spacetime supersymmetric spectrum. Throughout the calculations, we will only consider closed strings, and work only on the one-loop level in detail.

2.1 Spin Structures

A closed string vacuum energy (vacuum bubble) consists of several different loop contributions. The worldsheet of a g-loop vacuum bubble can be conformally mapped to a class of Riemann surfaces of genus g, *i.e.* a Riemann surface with g holes. This surface contains $2g$ uncontractable loops, and as now our theory contains fermions, we can impose either periodic $(+)$ or antiperiodic $(-)$ boundary conditions to our ψ field around each loop. One set of boundary conditions is called a *spin structure*. Therefore, on a Riemann surface of genus g, there exist 2^{2g} different spin structures. We also define the notion of even and odd spin structures: if the chiral Dirac operator (∇_z or $\nabla_{\bar{z}}$) has even (or odd) number of zero modes, then the spin structure is called even (or odd).

As we mentioned, we are only investigating the one-loop vacuum bubble in detail, whose worldsheet is conformally equivalent to a class of tori[11] (figure 1).

The torus can be parametrized by $\xi^1, \xi^2 \in [0,1]$, and conformally mapping it to the complex plane yields the complex coordinates $z = \xi^1 + \tau\xi^2$, and $\bar{z} = \xi^1 + \bar{\tau}\xi^2$. If we consider our original closed string which propagates in a time loop, then ξ^1 corresponds to the σ^1 space coordinate, and ξ^2 to the σ_2 time coordinate of the worldsheet. However, we secretly performed a Wick rotation,

[10] Bosons and fermions contribute with the opposite sign, which comes from the fact that the bosonic oscillators obey commutation relations, whereas the fermionic ones are anticommuting.

[11] A torus is a Riemann surface of genus 1.

so the time direction became a regular Euclidean direction as well. The τ is called the modular (or Teichmüller) parameter of the torus and it classifies conformally equivalent tori. Those tori, whose Teichmüller parameters can be transformed into each other by means of modular transformation are considered locally conformally equivalent. Recall the general modular transformation

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbb{R}. \tag{2.1}$$

Also recall the generators of modular transformations (Dehn twists), and how they transform the coordinates (ξ^1, ξ^2):

$$\begin{aligned} S : \tau \longrightarrow -1/\tau \quad &\Longrightarrow \left(\xi^1, \xi^2\right) \longrightarrow \left(-\xi^2, \xi^1\right), \\ T : \tau \longrightarrow \tau + 1 \quad &\Longrightarrow \left(\xi^1, \xi^2\right) \longrightarrow \left(\xi^1 + \xi^2, \xi^2\right). \end{aligned} \tag{2.2}$$

Separately for the right- and left-movers, there are four different spin structures on the torus. As usual, consider for now just the right-moving part.

$$\begin{array}{llll} (+,+): & \psi(\xi^1,\xi^2) = +\psi(\xi^1+1,\xi^2), & \psi(\xi^1,\xi^2) = +\psi(\xi^1,\xi^2+1) & \text{NS-sector} \\ (+,-): & \psi(\xi^1,\xi^2) = +\psi(\xi^1+1,\xi^2), & \psi(\xi^1,\xi^2) = -\psi(\xi^1,\xi^2+1) & \text{NS-sector} \\ (-,+): & \psi(\xi^1,\xi^2) = -\psi(\xi^1+1,\xi^2), & \psi(\xi^1,\xi^2) = +\psi(\xi^1,\xi^2+1) & \text{R-sector} \\ (-,-): & \psi(\xi^1,\xi^2) = -\psi(\xi^1+1,\xi^2), & \psi(\xi^1,\xi^2) = -\psi(\xi^1,\xi^2+1) & \text{R-sector} \end{array} \tag{2.3}$$

The periodicity or antiperiodicity in the first coordinate (ξ_1) corresponds to whether the right-moving part of our closed string belongs to the NS- or the R-sector. Which of these are even and which are odd? Fortunately, it is always possible to put a globally flat metric $\delta^{\alpha\beta}$ on the torus[12]. In this metric the chiral Dirac operator is simply ∂_z, and the only global zero mode for this is the constant spinor. If the spinor is constant, the boundary condition can only be $(+,+)$. So the only Dirac zero mode belongs to the $(+,+)$ spin structure, the others have none.

$$\begin{aligned} &(+,+) \rightarrow 1 \text{ zero mode: odd} \\ &(+,-) \rightarrow 0 \text{ zero mode: even} \\ &(-,+) \rightarrow 0 \text{ zero mode: even} \\ &(-,-) \rightarrow 0 \text{ zero mode: even} \end{aligned} \tag{2.4}$$

Why is the distinction between even and odd spin structures important? As we will see immediately in (2.5), the modular transformations (2.2) do not leave the spin structures invariant, they transform them into each other. These transformations, however, are reducible, the even and odd spin structures only transforming amongst themselves. See the Appendix C for further detail.

[12]Recall, that we did a Wick rotation, therefore we have the standard Euclidean metric, rather than the Minkowski $\eta^{\alpha\beta}$.

The spin structures on the torus transform in the following way under modular transformations:

$$
\begin{array}{llll}
S: & (+,+) \longrightarrow (+,+), & \begin{array}{l}(+,-) \longrightarrow (-,+)\\ (-,+) \longrightarrow (+,-)\\ (-,-) \longrightarrow (-,-)\end{array} \\[2ex]
T: & (+,+) \longrightarrow (+,+), & \begin{array}{l}(+,-) \longrightarrow (+,-)\\ (-,+) \longrightarrow (-,-)\\ (-,-) \longrightarrow (-,+)\end{array}
\end{array}
\tag{2.5}
$$

These can be proven fairly easily; one example is shown in Appendix D.

2.2 Partition Function

The partition function is basically the contribution of the vacuum bubbles. Recall the bosonic string one-loop partition function

$$
Z_{\text{bos}}(\tau) = \operatorname{Tr} e^{2\pi i \tau H}. \tag{2.6}
$$

Consider the generalization of this for the different spin structures of the right-moving part

$$
\begin{aligned}
Z^{++}(\tau) &= \eta_{++} \operatorname{Tr} e^{2\pi i \tau H_R} (-1)^F, \qquad & Z^{+-}(\tau) &= \eta_{+-} \operatorname{Tr} e^{2\pi i \tau H_R}, \\
Z^{-+}(\tau) &= \eta_{-+} \operatorname{Tr} e^{2\pi i \tau H_{NS}} (-1)^F, \qquad & Z^{--}(\tau) &= \eta_{--} \operatorname{Tr} e^{2\pi i \tau H_{NS}}.
\end{aligned}
\tag{2.7}
$$

Let us investigate step-by-step the differences compared to the bosonic string.

- Depending on the periodicity or antiperiodicity condition for ξ^1, we have the NS- or the R-sector, thus we need to use the corresponding light cone Hamiltonians

$$
\begin{aligned}
H_{NS} &= \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r - \frac{1}{6}, \\
H_R &= \sum_{m=1}^{\infty} m d^i_{-m} d^i_m + \frac{1}{3}.
\end{aligned}
\tag{2.8}
$$

  The normal ordering contributions follow from subtracting the bosonic part $-\frac{d-2}{24} = -\frac{1}{3}$ from the total normal ordering constant of the NS-sector ($a = -\frac{1}{2}$) or the R-sector ($a = 0$), respectively.

- The $(-1)^F$ contributions are needed when we have periodicity along ξ^2. It's not difficult, but quite lengthy to prove, that for anticommuting (Grassmann odd) variables the trace automatically ensures antiperiodicity condition along the time loop. Thus if we want to have periodic boundary conditions for states generated by odd number of fermionic creation oscillators, we need a (-1) multiplier. The product of even oscillators, however, is commuting, so we do not need such a factor. The prefactor $(-1)^F$ does precisely the job.

- In (2.5), we saw that modular transformations transform spin structures into each other. However, we want our theory to be invariant under local and global worldsheet diffeomorphisms, so we have to find a modular invariant combination of the partition functions. That is the purpose of the yet to be determined $\eta_{\pm\pm}$ phase factors.

Evaluation of the partition function contributions is now straightforward, although we have to remember the definitions of the Jacobi theta functions and Dedekind's eta function. As these functions are quite important, a quick recap can be found in Appendix E. With $q = e^{2\pi i\tau}$,

$$\begin{aligned} Z^{--}(\tau) &= \eta_{--}\,\mathrm{Tr}\,q^{H_{NS}} = \eta_{--}q^{-\frac{1}{6}}\,\mathrm{Tr}\,q^{\sum_{r=1/2}^{\infty}b^i_{-r}b^i_r} = \eta_{--}q^{-\frac{1}{6}}\prod_{r=1/2}^{\infty}\left[\sum_{N_r=0}^{1}q^{rN_r}\right]^8 = \\ &= \eta_{--}q^{-\frac{1}{6}}\left[\prod_{r=1/2}^{\infty}(q^r+1)\right]^8 = \eta_{--}q^{-\frac{1}{6}}\left[\prod_{n=1}^{\infty}\left(1+q^{n-1/2}\right)\right]^8 = \\ &= \eta_{--}\underbrace{\left[q^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)^{-1}\right]^4}_{\eta^{-1}(\tau)}\underbrace{\left[\prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n-1/2}\right)^2\right]^4}_{\theta_3(\tau)} = \eta_{--}\frac{\theta_3^4(\tau)}{\eta^4(\tau)}. \end{aligned} \tag{2.9}$$

In the first line, we used the definition (2.7), and took the anomaly out of the trace. We also made use of the fact that we have eight transverse modes, and, for each, the number operator appearing in the exponent can only take 0 or 1 values, as we are considering fermionic oscillators. The rest is just a shift in the product index r to integer values n, and then the identification of the emerging Jacobi theta and Dedekind's eta functions.

The partition functions for the other spin structure contributions can be evaluated similarly. The result is as follows.

$$\begin{aligned} Z^{++}(\tau) &= \eta_{++}\frac{\theta_1^4(\tau)}{\eta^4(\tau)}, \qquad & Z^{+-}(\tau) &= \eta_{+-}\frac{\theta_2^4(\tau)}{\eta^4(\tau)}, \\ Z^{-+}(\tau) &= \eta_{-+}\frac{\theta_4^4(\tau)}{\eta^4(\tau)}, \qquad & Z^{--}(\tau) &= \eta_{--}\frac{\theta_3^4(\tau)}{\eta^4(\tau)}. \end{aligned} \tag{2.10}$$

Examine what happens to the partition functions under modular transformations, using (E.9), which describes the modular transformation properties for the θ and η functions.

Under T transformations:

$$\begin{aligned} Z^{+-}(\tau+1) &= \eta_{+-}\frac{\theta_2^4(\tau+1)}{\eta^4(\tau+1)} = \eta_{+-}\frac{\theta_2^4(\tau)}{\eta^4(\tau)}e^{2i\pi/3} \xrightarrow{\cdot e^{-2i\pi/3}} \eta_{+-}\frac{\theta_2^4(\tau)}{\eta^4(\tau)}e^{2i\pi/3} = Z^{+-}(\tau), \\ Z^{-+}(\tau+1) &= \eta_{-+}\frac{\theta_4^4(\tau+1)}{\eta(\tau+1)} = \eta_{-+}\frac{\theta_3^4(\tau)}{\eta^4(\tau)}e^{-i\pi/3} \xrightarrow{\cdot e^{-2i\pi/3}} -\eta_{-+}\frac{\theta_3^4(\tau)}{\eta^4(\tau)} = -\frac{\eta_{--}}{\eta_{-+}}Z^{--}(\tau), \\ Z^{--}(\tau+1) &= \eta_{--}\frac{\theta_3^4(\tau+1)}{\eta(\tau+1)} = \eta_{--}\frac{\theta_4^4(\tau)}{\eta^4(\tau)}e^{-i\pi/3} \xrightarrow{\cdot e^{-2i\pi/3}} -\eta_{--}\frac{\theta_4^4(\tau)}{\eta^4(\tau)} = -\frac{\eta_{-+}}{\eta_{--}}Z^{-+}(\tau). \end{aligned} \tag{2.11}$$

Recall that the bosonic partition function $Z_{\text{bos}}(\tau) \propto {}^1\!/\!_{\eta^8(\tau)}$. Therefore, during T transformations, it acquires an $e^{-2i\pi/3}$ prefactor. Naturally, we have to include this as well, because the final partition function for a right-mover is the product of the bosonic and the fermionic partition functions.

$$Z(\tau) = Z_{\text{ferm}}(\tau) Z_{\text{bos}}(\tau) = \left[Z^{++}(\tau) + Z^{+-}(\tau) + Z^{-+}(\tau) + Z^{--}(\tau)\right] Z_{\text{bos}}(\tau) \tag{2.12}$$

Under S transformations:

$$\begin{aligned}
Z^{+-}(-{}^1\!/\!_\tau) &= \eta_{+-}\frac{\theta_2^4(-{}^1\!/\!_\tau)}{\eta^4(-{}^1\!/\!_\tau)} = \eta_{+-}\frac{\theta_4^4(\tau)}{\eta(\tau)} = \frac{\eta_{-+}}{\eta_{+-}} Z^{+-}(\tau), \\
Z^{-+}(-{}^1\!/\!_\tau) &= \eta_{-+}\frac{\theta_4^4(-{}^1\!/\!_\tau)}{\eta^4(-{}^1\!/\!_\tau)} = \eta_{-+}\frac{\theta_2^4(\tau)}{\eta(\tau)} = \frac{\eta_{+-}}{\eta_{-+}} Z^{+-}(\tau), \\
Z^{--}(-{}^1\!/\!_\tau) &= \eta_{--}\frac{\theta_3^4(-{}^1\!/\!_\tau)}{\eta^4(-{}^1\!/\!_\tau)} = \eta_{-+}\frac{\theta_3^4(\tau)}{\eta(\tau)} = Z^{--}(\tau).
\end{aligned} \tag{2.13}$$

As Z^{++} transforms irreducibly under modular transformations, we have not included it in (2.11) and (2.13) because requiring modular invariance would not help us determine the phase constant η_{++}. But for the other three, this is possible. For the odd spin structures, modular invariance under both T and S transformations is satisfied if we set

$$-\eta_{+-} = -\eta_{-+} = \eta_{--} \stackrel{!}{=} 1. \tag{2.14}$$

Only the relative phase matters between the constants, so we were free to set η_{--} to 1. The unknown η_{++} can be determined to be ± 1 by the following considerations. For Z^{++}, the partition function, to be interpreted as the sum over states, η_{++} can only take the values 0, 1, or -1. Consider now the two-loop partition function, and especially the limit where the genus 2 Riemann surface corresponding to it degenerates into two tori. In this limit, the genus 2 Jacobi theta functions can be expressed as products of the genus 1 functions. The degeneration limit of an even theta function of genus 2 is, for example, $\theta_1(\tau_1)\theta_1(\tau_2)$, where τ_1 and τ_2 are the Teichmüller parameters of the two tori, respectively. Even spin structures transform irreducibly under diffeomorphisms, and a genus 2 Riemann surface has ten of them. Thus, if η_{++} were zero, we would not have any of the even spin structures in the fermionic partition function, which is not possible. Therefore, the only possibilities left for this phase constant are to be

$$\eta_{++} = \pm 1. \tag{2.15}$$

We are now ready to write down the full partition function for the right-moving component of a closed string.

$$Z_{\text{ferm}}(\tau) = \operatorname{Tr} e^{2\pi i\tau H_{NS}} \underbrace{\frac{1}{2}\left(1-(-1)^F\right)}_{\text{GSO projection in the NS-sector}} - \operatorname{Tr} e^{2\pi i\tau H_R} \underbrace{\frac{1}{2}\left(1\pm(-1)^F\right)}_{\text{GSO projection in the R-sector}} = \\ = \frac{1}{2\eta^4(\tau)}\left[\underbrace{\theta_3^4(\tau)-\theta_4^4(\tau)-\theta_2^4(\tau)}_{\text{(E.8) Jacobi identity}=0} \pm \underbrace{\theta_1^4(\tau)}_{\theta_1\equiv 0^{15}}\right] = 0. \tag{2.16}$$

Therefore,

$$Z(\tau) = Z_{\text{ferm}}(\tau)Z_{\text{bos}}(\tau) = 0 \tag{2.17}$$

We notice that carefully considering the boundary conditions for Grassmann odd variables and requiring modular invariance yield precisely the GSO projection. Furthermore, we also see that, in the end, the partition function vanishes; the NS-sector (bosons) has the exact same contribution—albeit with opposite signs—as the R-sector (fermions). This is equivalent to saying that on each mass level we have the same number of bosons and fermions, which is a necessary condition for spacetime supersymmetry. The simple property of modular invariance justifies the seemingly arbitrary GSO conditions and provides us with a supersymmetric spectrum.

As a side note, we mention that only by requiring modular invariance separately for the right- and the left-movers do we get back the correct GSO projection. There is, however, another modular invariant choice, where the right- and left-movers are not separately modular invariant. This, however, does not interest us, as that theory only contains bosons, and even a tachyon.

[15] It is not surprising that that θ_1 is zero. From the physical point of view, we can understand it because the Dirac operator has a zero mode in the case of odd spin structure, so the path integral $Z^{++} \propto \int \mathcal{D}\,\psi e^{\psi \not{D} \psi} = 0$.

3 Green–Schwarz Formulation

In this chapter, we take a completely different approach and formulate a theory in a manifestly spacetime supersymmetric manner. After imposing some restrictions on the initial formulation, we will find that, thanks to the special property of the SO(8) group called *triality*, this manifest spacetime supersymmetric formulation is actually equivalent to the NSR formulation with the GSO projection.

As the Green–Schwarz (GS) formulation is quite complicated and requires a lot of meticulous and long calculations, we will not be as detailed in this chapter as in previous ones. Our main goal will be to understand the similarities and differences when compared to the NSR formulation and how the generated spectra of both theories are identical.

3.1 Spacetime Supersymmetry

Let us first briefly introduce what the concept of spacetime supersymmetry is. Basically, it is the generalization of the two-dimensional worldsheet supersymmetry introduced before to D spacetime dimensions. However, we will encounter some complications.

3.1.1 Point Particle

Consider first the following action of a point particle:

$$S = \frac{1}{2}\int d\tau\, e^{-1}(\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A)^2. \tag{3.1}$$

Compared to the regular bosonic contribution of $\dot{x}^2$, we introduced a term similar to the fermionic part of the NSR action (1.1). We say that our theory has N supersymmetries if it contains N Grassmann odd spinor variables θ^{Aa}, where $A = 1, \ldots, N$, and $a = 1, \ldots, 2^{[D/2]}$.

Besides the global Lorentz, the local reparametrization, and Weyl symmetries, this action is also invariant under three additional types of transformations.

Global supersymmetry: The symmetry transformation that relates the bosonic and the fermionic parts. With the anticommuting, infinitesimal spinor parameters ϵ^A, independent of position, these transformations take the form

$$\begin{aligned} \delta\theta^A &= \epsilon^A, \\ \delta x^\mu &= i\bar{\epsilon}^A\Gamma^\mu\theta^A, \\ \delta e &= 0. \end{aligned} \tag{3.2}$$

This is very similar to (1.3) of the NSR formulation.

The action (3.1) is, in fact, the simplest supersymmetric point particle action that we can come up with, as the quantity $p^\mu = \dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A$ is invariant under these transformations in itself.

Local fermionic symmetry: A local symmetry transformation with the help of the anticommuting, infinitesimal spinor $\kappa^A(\tau)$. It is a local symmetry, as now we allow the infinitesimal parameter to depend on the worldline position.

$$\begin{aligned}
\theta^A &= i\Gamma \cdot p\kappa^A, \\
\delta x^\mu &= i\bar{\theta}^A\Gamma^\mu\delta\theta^A, \\
\delta e &= 4e\dot{\bar{\theta}}^A\kappa^A.
\end{aligned} \tag{3.3}$$

Local bosonic symmetry: A third additional local symmetry with the scalar parameter $\lambda(\tau)$.

$$\begin{aligned}
\delta\theta^A &= \lambda\dot{\theta}^A, \\
\delta x^\mu &= i\bar{\theta}^A\Gamma^\mu\delta\theta^A, \\
\delta e &= 0.
\end{aligned} \tag{3.4}$$

3.1.2 Strings

The generalization of the point particle action (3.1) is far from trivial.

$$S = S_1 + S_2,$$

$$\begin{aligned}
S_1 &= -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{h}h^{\alpha\beta}\Pi_\alpha\cdot\Pi_\beta, \qquad \text{where} \quad \Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A, \\
S_2 &= \frac{1}{\pi}\int d^2\sigma\,\left\{-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + \epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\right\}.
\end{aligned} \tag{3.5}$$

The obvious generalization would only consist of the S_1 part, but it can be calculated that it no longer carries the local κ symmetry, which we would like to keep. This symmetry can be restored by the addition of the term S_2, but only for $N = 0, 1, 2$ superymmetries[16].

However, now it is not obvious at all that S_2 separately carries all the symmetries that S_1 does. In fact, it generally does not. It clearly has the global Lorentz, the local reparametrization symmetries, and, as it does not even depend on $h_{\alpha\beta}$, the Weyl scaling invariance is obviously satisfied as well. On the other hand, requiring global supersymmetry of S_2 constrains our theory to the following four cases:

- $D = 3$ and θ is Majorana,
- $D = 4$ and θ is Majorana or Weyl,
- $D = 6$ and θ is Weyl,
- $D = 10$ and θ is Majorana–Weyl.

[16] We will always show the formulas for the $N = 2$ case; the others are easily achieved by setting one or both θ coordinates to zero.

Our interest lies only in the last case, because we have already seen that the quantization of a supersymmetric string theory will restrict the spacetime dimensions to $D = 10$.

By investigating the algebra of the local κ transformations, one will find that its closure requires another local bosonic symmetry to be present. Thus, our generalized superstring action also bears all the symmetries found for the point particle.

3.2 Supersymmetric Theories

As we have previously discussed, the Majorana and the Weyl coniditions both halve the number of independent components for θ^1 and θ^2, reducing it from the original 32 complex to 16 real, for both of them. Furthermore, the Weyl condition implies definite chirality for both θ fields, so the relation between them will give rise to different theories.

Type I theory: This theory involves only open superstrings. We will see a bit later that to keep as many supersymmetries as possible, one has to equate the two θ fields at the end of the string. If $\theta^1 = \theta^2$ at any point of the string, they cannot have different chiralities. Therefore, for open strings, there is only one theory, where θ^1 and θ^2 has the same handedness. Even with this, the maximum supersymmetry that we can keep is $N = 1$, hence the naming type I.

Type IIA theory: A closed superstring theory where θ^1 and θ^2 have opposite handedness. In this case, it is possible to keep $N = 2$ supersymmetry, *i.e.* this is a type II theory.

Type IIB theory: A closed superstring theory where θ^1 and θ^2 have the same chirality. As the naming suggests, it also has $N = 2$ supersymmetry.

Heterotic string theory: A closed string theory where we only use one θ coordinate. The right-moving part is a superstring whereas the left-moving part is a bosonic string.

Our focus will mainly lie on the type II theories, and it is not a coincidence that these theories bear the same name as those of the NSR formulation. At the end of the day, we will find that the GS and the NSR formulations are equivalent.

3.2.1 Equivalence of the GS and NSR Formulations

Using the Weyl and reparameztrization invariances, we can fix $h_{\alpha\beta} = \eta_{\alpha\beta}$ as usual and, without discussing it in much detail, we can enforce the light cone gauge with the help of the remaining symmetries. In the end we are left with eight degrees of freedom for both θ^1 and θ^2, which can be viewed as a spinor of SO(8), due to the light cone gauge still posessing global SO(8) rotational symmetry.

We use a new symbol for the remaining eight degrees of freedom of θ^1 and θ^2: S^1 and S^2. As these spinors are Weyl, conventionally, we say that S^1 belongs to $\mathbf{8_s}$. So, for type I and type IIB theories, S^2 is also $\mathbf{8_s}$, but for the type IIA theory, S^2 is a $\mathbf{8_c}$ conjugate spinor.

Skipping the details, it can be shown that from the lengthy and complicated action (3.5), the following equations of motion can be derived after gauge fixing:

$$\begin{aligned}\left(\frac{\partial^2}{\partial\sigma^2}-\frac{\partial^2}{\partial\tau^2}\right)X^i&=0,\\ \left(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\tau}\right)S^{1a}&=0,\\ \left(\frac{\partial}{\partial\sigma}-\frac{\partial}{\partial\tau}\right)S^{2a}&=0.\end{aligned}\tag{3.6}$$

These equations of motion can also be derived from a much simpler looking action, which we will call the Green–Schwarz action in light cone gauge.

$$S_{GS}^{l.c.}=-\frac{1}{2\pi}\int d^2\sigma\,(\partial_\alpha X^i\partial^\alpha X^i-i\bar{S}^a\rho^\alpha\partial_\alpha S^a).\tag{3.7}$$

Here, S^1 and S^2 were combined into a two-component Majorana worldsheet spinor S. Separately, they are one-component Majorana–Weyl spinors on the worldsheet, describing right- or left-movers. This action is extremely similar to the light cone gauge NSR action (1.4). There is no difference in the bosonic part: X^i is an $\mathbf{8_v}$ representation of SO(8) in both cases. However, while the fermionic parts, ψ^i and S^a, are both two-component Majorana worldsheet spinors, they are of different SO(8) representations. ψ^i is an $\mathbf{8_v}$ (just like X^i) for both movers, but S^{1a} is an $\mathbf{8_s}$ for the right-movers and S^{2a} is an $\mathbf{8_{s/c}}$ for the left-movers (depending on which theory we are discussing).

This is the fundamental difference between the NSR and the GS formulations and, as mentioned before, there exists a special relation unique to SO(8) called triality, which can permute these representations. This results in the fact that the NSR formulation (with the GSO projection) and the GS formulation are equivalent. ψ^i and S^a can be transformed into each other by means of bosonization, shuffling of the acquired bosonic coordinates, and finally refermionizing them, as shown below.

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_1=\bar{\psi}^1\rho^\alpha\psi^2, &&\sigma_1=\frac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4), &&\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_1=\bar{S}^1\rho\alpha S^2,\\ &\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_2=\bar{\psi}^3\rho^\alpha\psi^4, &&\sigma_2=\frac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4), &&\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_2=\bar{S}^3\rho\alpha S^4,\\ &\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_3=\bar{\psi}^5\rho^\alpha\psi^6, &&\sigma_3=\frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4), &&\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_3=\bar{S}^5\rho\alpha S^6,\\ &\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_4=\bar{\psi}^7\rho^\alpha\psi^8, &&\sigma_4=\frac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4), &&\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_4=\bar{S}^7\rho\alpha S^8.\end{aligned}\tag{3.8}$$

3.2.2 Quantization and the Spectrum

Quantization is exactly the same for the bosonic part (so we shall not repeat it) and the fermionic part is very similar to that of the NSR formulation, only using spinor indices a (or $\bar{a}$)

instead of vector ones. The anticommutation relation reads

$$\left\{S^{Aa}(\sigma,\tau),S^{Bb}(\sigma',\tau)\right\}=\pi\delta^{ab}\delta^{AB}\delta(\sigma-\sigma'). \tag{3.9}$$

However, if we are not careful, the boundary conditions can destroy the manifest spacetime supersymmetries, something that we would not want. Therefore, we do not have as much freedom as in the NSR formulation, where we could define NS- and R-sectors depending on different boundary conditions. Here, we always have to equate the right- and left-moving parts at the boundaries for open strings, and can only impose periodicity for both movers for closed strings. For closed strings, we can keep $N=2$, but for open strings, the maximal supersymmetry we can keep is $N=1$[17].

$$\begin{array}{ll} \text{Open strings (type I)} & \text{Closed strings (type II)} \\ S^{1a}(0,\tau)=S^{2a}(0,\tau), & S^{1a}(\sigma,\tau)=S^{1a}(\sigma+\pi,\tau), \\ S^{1a}(\pi,\tau)=S^{2a}(\pi,\tau). & S^{2a}(\sigma,\tau)=S^{2a}(\sigma+\pi,\tau). \end{array} \tag{3.10}$$

Therefore the oscillator expansions read

$$\begin{array}{ll} S^{1a}=\dfrac{1}{\sqrt{2}}\sum S^a_n e^{-in(\tau-\sigma)}, & S^{1a}(\sigma,\tau)=\sum S^a_n e^{-2in(\tau-\sigma)}, \\ S^{1a}=\dfrac{1}{\sqrt{2}}\sum S^a_n e^{-in(\tau+\sigma)}. & S^{2a}(\sigma,\tau)=\sum \tilde{S}^a_n e^{-2in(\tau+\sigma)}. \end{array} \tag{3.11}$$

Reality (Majorana) condition implies

$$S^a_{-m}=\left(S^a_m\right)^\dagger. \tag{3.12}$$

Also, the canonical anticommutation relation is

$$\left\{S^a_m,S^b_n\right\}=\delta^{ab}\delta_{m+n}. \tag{3.13}$$

The mass-squared operator is defined as

$$\alpha' m^2=\underbrace{\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n}_{N^{(\alpha)}}+\underbrace{\sum_{n=1}^{\infty}nS^a_{-n}S^a_n}_{N^{(S)}}. \tag{3.14}$$

We have no normal ordering constant in (3.14), which implies that the ground state is massless. Also, as seen in the R-sector of the NSR formulation, due to the presence of S_0, we have a ground state degeneracy. Because of (3.13), S_0 satisfies an algebra similar to the Clifford algebra. The difference is that we have now spinor instead of vector indices:

$$\left\{S^a_0,S^b_0\right\}=\delta^{ab}. \tag{3.15}$$

[17] The boundary condition will imply that the supersymmetry transformation of S^1 and S^2 will require the same infinitesimal $\delta\epsilon$ parameter for open strings.

$\alpha' m^2$	states and their SO(8) representation contents		little group	representation contents with respect to the little group
0	$\vert i\rangle$ $\mathbf{8_v}$ $\vert\bar{a}\rangle$ $\mathbf{8_c}$		SO(8)	$\mathbf{8_v}$ (boson) $\mathbf{8_c}$ (fermion)
+1	$\alpha^j_{-1}\,\vert i\rangle$ $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$ $\alpha^i_{-1}\,\vert\bar{a}\rangle$ $\mathbf{8_s} \oplus \mathbf{56_s}$	$S^b_{-1}\,\vert\bar{a}\rangle$ $\mathbf{8_v} \oplus \mathbf{56_v}$ $S^b_{-1}\,\vert i\rangle$ $\mathbf{8_c} \oplus \mathbf{56_c}$	SO(9)	$\mathbf{84} \oplus \mathbf{44}$ (bosons) $\mathbf{128}$ (fermions)

Table 3: The spectrum until the first massive level in the GS formulation, built on a $\mathbf{8_v} \oplus \mathbf{8_c}$ ground state.

Fortunately, as also mentioned in Appendix B, triality helps us. The construction of the ground state in the GS formulation is exactly the same as in the R-sector of the NSR formulation, but with $\mathbf{8_v} \leftrightarrow \mathbf{8_{s/c}}$. This means that we can just redefine the Dirac matrices with indices i and a (or $\bar{a}$) exchanged.

$$S_0^a \propto \gamma^a = \begin{pmatrix} 0 & \gamma^a_{i\,\bar{a}} \\ \gamma^a_{\bar{a}i} & 0 \end{pmatrix}, \qquad \text{or} \qquad S_0^{\bar{a}} \propto \gamma^{\bar{a}} = \begin{pmatrix} 0 & \gamma^{\bar{a}}_{i\,a} \\ \gamma^{\bar{a}}_{ai} & 0 \end{pmatrix}. \tag{3.16}$$

Correspondingly, the ground state is now not an $\mathbf{8_s} \oplus \mathbf{8_c}$ multiplet, but an $\mathbf{8_v} \oplus \mathbf{8_{c/s}}$ one.

$$|\phi_0\rangle_{\mathbf{8_v}\oplus\mathbf{8_c}} = |i\rangle\,\zeta^i(k) + |\bar{a}\rangle\,\lambda_c^{\bar{a}}(k), \qquad \text{or} \qquad |\phi_0\rangle_{\mathbf{8_v}\oplus\mathbf{8_s}} = |i\rangle\,\zeta^i(k) + |a\rangle\,\lambda_s^a(k). \tag{3.17}$$

The right-mover (open string) spectrum built on this ground state until the first massive level is shown in Tables [3] and [4].

The equivalence of the NSR and GS formulations can also be seen in practice by comparing Tables [3] and [4] with Table [1]. The two different tables, built on two different ground state multiplets in the GS formulation, correspond to the two choices we have in the GSO projection in the NSR formulation ($\bar{\Gamma} = +1$ or -1).

Closed superstring states are again tensor products of right- and left-movers.

Type IIA: Opposite chirality for right- and left-movers.

$$\begin{aligned}(\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) = &(\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \oplus \mathbf{8_v} \oplus \mathbf{56_v})_{\text{bosonic}} \\ &\oplus (\mathbf{8_s} \oplus \mathbf{8_c} \oplus \mathbf{56_s} \oplus \mathbf{56_c})_{\text{fermionic}}.\end{aligned} \tag{3.18}$$

$\alpha' m^2$	states and their SO(8) representation contents	little group	representation contents with respect to the little group
0	$\|i\rangle$ $\mathbf{8_v}$ $\|a\rangle$ $\mathbf{8_s}$	SO(8)	$\mathbf{8_v}$ (boson) $\mathbf{8_s}$ (fermion)
+1	$\alpha^j_{-1}\|i\rangle$ $\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}$; $S^{\bar b}_{-1}\|a\rangle$ $\mathbf{8_v}\oplus\mathbf{56_v}$ $\alpha^i_{-1}\|a\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$; $S^{\bar b}_{-1}\|i\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$	SO(9)	$\mathbf{84}\oplus\mathbf{44}$ (bosons) $\mathbf{128}$ (fermions)

Table 4: The spectrum until the first massive level in the GS formulation, built on a $\mathbf{8_v}\oplus\mathbf{8_s}$ ground state.

Type IIB: Same chirality for the right- and left-movers.

$$\begin{aligned}(\mathbf{8_v}\oplus\mathbf{8_c})\otimes(\mathbf{8_v}\oplus\mathbf{8_c}) = &(\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}\oplus\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_c})_{\text{bosonic}} \\ &\oplus(\mathbf{8_s}\oplus\mathbf{8_s}\oplus\mathbf{56_s}\oplus\mathbf{56_s})_{\text{fermionic}}.\end{aligned} \tag{3.19}$$

Comparing formulas (3.18) with (1.30), and (3.19) with (1.31), we see that both formulations give the same spectrum for closed strings as well.

References

In the course of writing the presentation and this report for the proseminar, I mainly relied on the following resource materials:

1. M.B. Green, J.H. Schwarz, E. Witten, Superstring theory I & II, Cambridge University Press (1987) and (1988).

2. D. Lüst, S. Theisen, Lectures on string theory, Lecture notes in physics, Springer (1989).

Also, I would like to thank my supervisor, Cristian Vergu, for always being really helpful in explaining the various new concepts one comes across when starting to study string theory.

Appendices

A Dirac Equation for Weyl Spinors

Let us derive how we can obtain (1.24) just for positive chirality (the negative one can be done analogously). As we saw in (1.21), the chirality operator takes the form

$$\Gamma_{11} = -\sigma_1 \otimes \begin{pmatrix} \mathbb{1}_8 & \\ & -\mathbb{1}_8 \end{pmatrix}. \tag{A.1}$$

We will start with the spinor $\chi = \alpha \otimes \lambda$, where α is two-component vector, and $\lambda = (\lambda_s^a, \lambda_c^{\bar{a}})$ is an SO(8) spinor. In the calculations, we absorb all worldsheet coordinate dependencies into the λ part.

$$\Gamma_{11}\chi = \left[-\sigma_1 \otimes \begin{pmatrix} \mathbb{1}_8 & \\ & -\mathbb{1}_8 \end{pmatrix}\right] \chi = \sigma_1 \alpha \otimes \begin{pmatrix} \mathbb{1}_8 & \\ & -\mathbb{1}_8 \end{pmatrix} \lambda. \tag{A.2}$$

To obtain +1 eigenvalue for the chirality operator (positive chirality), we have two possibilities. Either α and λ are both positive-eigenvalued eigenvectors for $-\sigma_1$ and $\begin{pmatrix} \mathbb{1}_8 & \\ & -\mathbb{1}_8 \end{pmatrix}$, respectively, or they are both negative. The eigenvectors are:

$$\begin{aligned} &(++): \qquad -\sigma_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} \mathbb{1}_8 & \\ & -\mathbb{1}_8 \end{pmatrix} \begin{pmatrix} \lambda_s \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_s \\ 0 \end{pmatrix}. \\ &(--): \qquad -\sigma_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \mathbb{1}_8 & \\ & -\mathbb{1}_8 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_c \end{pmatrix} = -\begin{pmatrix} 0 \\ \lambda_c \end{pmatrix}. \end{aligned} \tag{A.3}$$

χ is the linear combination of these two cases.

$$\chi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} \lambda_s \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \lambda_c \end{pmatrix}. \tag{A.4}$$

The Dirac equation thus reads:

$$\begin{aligned} 0 &= i\Gamma^\mu \partial_\mu \chi = i\Gamma^0 \partial_0 \chi + i\Gamma^i \partial_i \chi + i\Gamma^9 \partial_9 \chi = i\sigma_2 \otimes \mathbb{1}_{16} \partial_0 \chi + ii\sigma_1 \otimes \gamma^i \partial_i \chi + ii\sigma_3 \otimes \mathbb{1}_{16} \partial_9 \chi = \\ &= \left\{ (i\sigma_2 \otimes \mathbb{1}_{16})\, \partial_0 - (\sigma_1 \otimes \gamma^i)\, \partial_i - (\sigma_3 \otimes \mathbb{1}_{16})\, \partial_9 \right\} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} \lambda_s \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \lambda_c \end{pmatrix} \right] = \\ &= \left[-\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \partial_0 \lambda_s \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \partial_0 \lambda_c \end{pmatrix} \right] - \left[-\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \gamma^i_{\bar{a}a} \partial_i \lambda^a_s \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \gamma^i_{a\bar{a}} \partial_i \lambda^{\bar{a}}_c \\ 0 \end{pmatrix} \right] + \\ &- \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \partial_9 \lambda_s \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \partial_9 \lambda_c \end{pmatrix} \right] = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} (\partial_0 + \partial_9)\lambda_s \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \partial_i \gamma^i_{a\bar{a}} \lambda^{\bar{a}}_c \\ 0 \end{pmatrix} + \\ &+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ (\partial_0 - \partial_9)\lambda_c \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \gamma^i_{\bar{a}a} \lambda^a_s \end{pmatrix}. \end{aligned}$$

As $(1,1)$ and $(1,-1)$ are linearly independent, the expressions belonging to those vectors must equal zero separately, thereby reaching (1.24).

B Representation Contents

This appendix helps in explaining what the meanings of the numbers and their small indices in Tables [1–4] are.

B.1 SO(8) Representation Contents

SO(8) has three fundamental representations: vector ($\mathbf{8_v}$), spinor ($\mathbf{8_s}$), and conjugate spinor ($\mathbf{8_c}$), as mentioned before. The number tells us about the number of degrees of freedom we have in each. When applying a creation operator to a certain state, we reach another one with different representation contents. In the NSR formulation, with the light cone gauge fixed, all operators belong to the $\mathbf{8_v}$ representation of SO(8). Thus, in terms of representation theory, applying these operators means taking the tensor product of an $\mathbf{8_v}$ vector with the corresponding state. In the GS formulation, there are operators belonging to each of the three fundamental SO(8) representations. Thus, it is useful to see how the bilinears belonging to the tensor product of these representations can be decomposed into irreducible parts, because they correspond to fundamentally different objects/particles.[18]

First, let us start with the easy one. A bilinear of two vectors is a rank-2 tensor, which decomposes into a trace part, an antisymmetric part, and a symmetric traceless part. Each of these transform irreducibly under SO(8) rotations.

$$\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \frac{\mathbf{8 \cdot 7}}{\mathbf{2}} \oplus \left(\frac{\mathbf{8 \cdot 9}}{\mathbf{2}} - \mathbf{1} \right) = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}. \tag{B.1}$$

Not all of these are always present. For example, in the NSR formulation, the α oscillators obey commutation, and the b and d oscillators obey anticommutation relations, so the bilinears formed from the same oscillators keep the symmetric or the antisymmetric parts, respectively. When a bilinear is formed from two different oscillators, however, there are no restrictions: all parts are present.

Next, consider the bilinear formed by a spinor and a conjugate spinor. This is a bit more complicated. First, let us note that for 16×16 matrices, the following combinations of the Dirac matrices form a complete basis:

$$\begin{array}{ll} \mathbb{1}_{16}, & \gamma^{ijklmnop}, \\ \gamma^{i}, & \gamma^{ijklmno}, \\ \gamma^{ij}, & \gamma^{ijklmn}, \\ \gamma^{ijk}, & \gamma^{ijklm}, \\ \multicolumn{2}{c}{\gamma^{ijkl},} \end{array} \tag{B.2}$$

[18] Lorentz transformation, or, more precisely, just the SO(8) rotation cannot transform these parts into each other.

where $\gamma^{i_1 i_2 \dots i_n} = \gamma^{[i_1}\gamma^{i_2}\dots\gamma^{i_n]}$, the antisymmetrized product of gamma matrices. Therefore, any bilinear can be decomposed on this basis.

$$\lambda_a \omega_{\bar{a}} = \sum_p \left(\gamma^{i_1 i_2 \dots i_p}\right)_{a\bar{a}} \varphi_{i_1 i_2 \dots i_p}, \tag{B.3}$$

where $\varphi_{i_1 i_2 \dots i_p}$ are just coefficients. Recall the expression (1.17) for the gamma matrices, and especially the fact that they are off-diagonal. The product of even gamma matrices is, therefore, diagonal, and the product of odd ones becomes off-diagonal again. When trying to find the decomposition of a spinor–conjugate spinor bilinear, we are only interested in the off-diagonal elements, due to the index structure, as the index pair $a\bar{a}$ clearly refers to the off-diagonal elements. Therefore, we only have to consider the product of odd number of gamma matrices in the decomposition.

We are always considering Weyl spinors, which will simplify things slightly, as, for Weyl spinors, it is not difficult to prove that the following duality relations hold amongst the gamma matrices:

$$\begin{aligned}
\mathbb{1}_{16} \quad &\propto \quad \epsilon^{ijklmnop}\gamma_{ijklmnop}, \\
\gamma^{i} \quad &\propto \quad \epsilon^{ijklmnop}\gamma_{jklmnop}, \\
\gamma^{ij} \quad &\propto \quad \epsilon^{ijklmnop}\gamma_{klmnop}, \\
\gamma^{ijk} \quad &\propto \quad \epsilon^{ijklmnop}\gamma_{lmnop}, \\
\gamma^{ijkl} \quad &\propto \quad \epsilon^{ijklmnop}\gamma_{mnop}. \qquad \text{(self-duality!)}
\end{aligned} \tag{B.4}$$

Therefore, the only relevant degrees of freedom come from γ^i and γ^{ijk}. For the latter, the number of independent components can be easily calculated by the fact that it is fully antisymmetric.

$$\mathbf{8_s} \otimes \mathbf{8_c} = \mathbf{8_v} \oplus \frac{\mathbf{8 \cdot 7 \cdot 6}}{\mathbf{2 \cdot 3}} = \mathbf{8_v} \oplus \mathbf{56_v}. \tag{B.5}$$

For a spinor–spinor bilinear, the procedure is the same, but we need the diagonal elements of the product of gamma matrices, as both indices now are spinor indices, which correspond to the diagonal positions. Therefore, the only relevant matrices in the decomposition are now $\mathbb{1}_{16}$, γ^{ij}, and γ^{ijkl}, the even products. The number of degrees of freedom for γ^{ijkl} is reduced by a factor of two because it is self-dual.

$$\mathbf{8_s} \otimes \mathbf{8_s} = \mathbf{1} \oplus \frac{\mathbf{8 \cdot 7}}{\mathbf{2}} \oplus \frac{\mathbf{1}}{\mathbf{2}} \cdot \frac{\mathbf{8 \cdot 7 \cdot 6 \cdot 5}}{\mathbf{2 \cdot 3 \cdot 4}} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}. \tag{B.6}$$

For a conjugate spinor–conjugate spinor bilinear, this is done exactly the same way. All that is left now is to examine the decomposition of the spinor/conjugate spinor–vector bilinear. We can make use of the special concept called triality unique to SO(8). Without much detail, triality states that all three fundemental representations of the SO(8) group are “equally good” or, more precisely, there exists a group of automorphisms that permute these representations. So the procedure in examining the remaining bilinears is basically again the same, we just redefine the gamma matrices used in the decomposition to have their indices $i \leftrightarrow a$ or $i \leftrightarrow \bar{a}$ exchanged. The results are very similar to the ones discussed above[19].

So let us summarize the decomposition rules.

$$\begin{aligned} \mathbf{8_v} \otimes \mathbf{8_v} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}, \qquad & \mathbf{8_s} \otimes \mathbf{8_c} &= \mathbf{8_v} \oplus \mathbf{56_v}, \\ \mathbf{8_s} \otimes \mathbf{8_s} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_s}, & \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_c}, \\ \mathbf{8_c} \otimes \mathbf{8_c} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c}, & \mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_s}. \end{aligned} \tag{B.7}$$

If a state belongs to a vector representation, then it is a boson; if it belongs to a spinor or a conjugate spinor representation, it is a fermion. Looking at (B.7), it is clear now why operators that belong to the vector representation of SO(8) can only map bosons to bosons and fermions to fermions. This is why, in the NSR formulation, the ground state determines whether the entire spectrum built on it is fermionic or bosonic. However, in the GS formulation, the operator of the fermionic part belongs to the spinor or conjugate spinor representation of SO(8), so it maps between bosons and fermions.

B.2 SO(9) Representation Contents

For massive states, the SO(8) representations can be uniquely assembled into SO(9) representations. Again, without detailed proof, we will just give examples corresponding to Table [1] so that the numbers in the last column make sense.

NS-sector: Form a bilinear (rank-2 tensor) of two vector representations of SO(9). Just as we have seen for the SO(8) case, this decomposes into the trace, the antisymmetric, and the symmetric traceless parts.

$$\mathbf{9_v} \otimes \mathbf{9_v} = \mathbf{1} \oplus \mathbf{36} \oplus \mathbf{44}. \tag{B.8}$$

The trace corresponds to the ground state of the NS-sector. The antisymmetric and the symmetric traceless part contain some of the SO(8) representations.

$$A^{IJ}: \quad (I, J = 1, \ldots, 9) \quad \mathbf{36} \quad \left\{ \begin{array}{lll} A^{i9}: & (i = 1, \ldots, 8) & \mathbf{8_v} \\ A^{ij}: & (i, j = 1, \ldots, 8) & \mathbf{28} \end{array} \right. \tag{B.9}$$

$$S^{IJ}: \quad (I, J = 1, \ldots, 9) \quad \mathbf{44} \quad \left\{ \begin{array}{lll} S^{99}: & & \mathbf{1} \\ S^{i9}: & (i = 1, \ldots, 8) & \mathbf{8_v} \\ S^{ij}: & (i, j = 1, \ldots, 8) & \mathbf{35_v} \end{array} \right. \tag{B.10}$$

Also look at the decomposition of a vector–vector–vector trilinear (rank-3 tensor):

$$\mathbf{9_v} \otimes \mathbf{9_v} \otimes \mathbf{9_v} = \mathbf{1} \oplus \mathbf{84} \oplus \mathbf{644}. \tag{B.11}$$

[19] In fact, we can also get the decomposition of a vector–vector bilinear this way, though the previous considerations are much simpler.

$$A^{IJK}: \quad (I,J,K=1,\ldots,9) \quad \mathbf{84} \quad \left\{ \begin{array}{lll} A^{ij9}: & (i,j=1,\ldots,8) & \mathbf{28} \\ A^{ijk}: & (i,j,k=1,\ldots,8) & \mathbf{56} \end{array} \right. \tag{B.12}$$

These are exactly the numbers we indicated in the NS-sector of the open string spectrum in Table [1].

R-sector: A spinor of SO(9) can also be represented by an array of $2^4 = 16$ complex elements[20], which again consists of a spinor ($\mathbf{8_s}$) and a conjugate spinor ($\mathbf{8_c}$) part. Majorana condition is possible[21], so let us impose it, reducing the number of degrees of freedom to 16 real. Take an SO(9) vector ζ^I ($I = 1,\ldots,9$), and a spinor $\lambda = (\lambda_s^a, \lambda_c^{\bar{a}})$, and examine the bilinear formed from these two.

$$\mathbf{9_v} \otimes (\mathbf{8_s} \oplus \mathbf{8_c}) = \mathbf{16} \oplus \mathbf{128}. \tag{B.13}$$

The tensor product decomposes into two irreducible parts. The **16** corresponds to the spinor trace defined as $\tau_c = \Gamma^I_{cd}\zeta_I\lambda^d$, and the **128** part is just the rest, which also appears in Table [1].

C Spin Structures in Higher Genus Riemann Surfaces

Consider the following two statements without proof:

1. For a given spin structure, the number of chiral Dirac zero modes is a topological invariant modulo two.

2. The number of chiral Dirac zero modes is additive modulo two when gluing together two Riemann surfaces.

First of all, the consequence of the first statement is that the even and odd spin structures transform separately (in fact, irreducibly) under modular transformations, just as we said in Chapter 2.1. The second statement, and our result for the torus (1 odd, 3 even spin structures), is enough to find the number of even and odd spin structures for higher genus Riemann surfaces.

$$\begin{aligned} \#\text{odd} &= \sum_{m \text{ odd}} \binom{g}{m} 1^m 3^{g-m} = 2^{g-1}(2^g - 1), \\ \#\text{even} &= \sum_{m \text{ even}} \binom{g}{m} 1^m 3^{g-m} = 2^{g-1}(2^g + 1). \end{aligned} \tag{C.1}$$

[20] Recall that a spinor in D dimensions has $2^{[D/2]}$ components.

[21] Here we are talking about a spinor of SO(D) and not of SO(1,$D-1$), so there is no conflict with our previous statement on the possible dimensions where the reality condition can be imposed.

D Spin Structures under Modular Transformations

Here, we will examine as an example, how the spin structure $(-,+)$ on the torus transforms under the generators of modular transformations. The transformation rules for the others can be derived analogously.

$(-,+)$ means the following:

$$\psi(\xi^1,\xi^2) = -\psi(\xi^1+1,\xi^2), \tag{D.1}$$
$$\psi(\xi^1,\xi^2) = +\psi(\xi^1,\xi^2+1). \tag{D.2}$$

The coordinates transform according to (2.2), so we can easily derive the change in the boundary conditions.

Under S transformations:

$$\begin{array}{ccc} \psi(\xi^1,\xi^2) & \stackrel{(D.1)}{=} & -\;\psi(\xi^1+1,\xi^2) \\ \downarrow & & \downarrow \\ \psi(-\xi^2,\xi^1) & \stackrel{(D.2)}{=} & +\;\psi(-\xi^2,\xi^1+1) \\ & & \\ \psi(\xi^1,\xi^2) & \stackrel{(D.2)}{=} & +\;\psi(\xi^1,\xi^2+1) \\ \downarrow & & \downarrow \\ \psi(-\xi^2,\xi^1) & \stackrel{(D.1)}{=} & -\;\psi(-\xi^2-1,\xi^1) \end{array} \tag{D.3}$$

Under T transformations:

$$\begin{array}{ccc} \psi(\xi^1,\xi^2) & \stackrel{(D.1)}{=} & -\;\psi(\xi^1+1,\xi^2) \\ \downarrow & & \downarrow \\ \psi(\xi^1+\xi^2,\xi^2) & \stackrel{(D.1)}{=} & -\;\psi(\xi^1+\xi^2+1,\xi^2) \\ & & \\ \psi(\xi^1,\xi^2) & \stackrel{(D.2)}{=} & +\;\psi(\xi^1,\xi^2+1) \\ \downarrow & & \downarrow \\ \psi(\xi^1+\xi^2,\xi^2) & \underset{(D.2)}{\stackrel{(D.1)}{=}} & -\;\psi(\xi^1+\xi^2+1,\xi^2+1) \end{array} \tag{D.4}$$

The change in the signs is apparent.

$$S: \quad (-,+) \longrightarrow (+,-), \qquad \text{and} \qquad T: \quad (-,+) \longrightarrow (-,-)$$

E Jacobi Theta Functions

The general form of the Jacobi theta function is the following ($q = e^{2\pi i\tau}$):

$$\theta\begin{bmatrix}\vartheta\\ \varphi\end{bmatrix}(0|\tau) = \eta(\tau)e^{2\pi i\vartheta\varphi}q^{\frac{\vartheta^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+q^{n+\vartheta-1/2}e^{2\pi i\varphi}\right)\left(1+q^{n-\vartheta-1/2}e^{-2\pi i\varphi}\right) = \\ = \sum_{n=-\infty}^{\infty}\exp\left[i\pi(n+\vartheta)^2\tau + 2\pi i(n+\vartheta)\varphi\right]. \tag{E.1}$$

We will only use those where $\vartheta = \varphi = 0, \frac{1}{2}$.

$$\theta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(0|\tau) = \theta_1(\tau) \equiv 0, \tag{E.2}$$

$$\theta\begin{bmatrix}1/2\\ 0\end{bmatrix}(0|\tau) = \theta_2(\tau) = 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)\left(q^n\right)^2 = \sum_{n=-\infty}^{\infty}q^{(n+1/2)^2/2}, \tag{E.3}$$

$$\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0|\tau) = \theta_3(\tau) = \prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n-1/2}\right)^2 = \sum_{n=-\infty}^{\infty}q^{\frac{n^2}{2}}, \tag{E.4}$$

$$\theta\begin{bmatrix}0\\ 1/2\end{bmatrix}(0|\tau) = \theta_4(\tau) = \prod_{n=1}^{\infty}(1-q^n)\left(1-q^{n-1/2}\right)^2 = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n^2}{2}}. \tag{E.5}$$

Dedekind's eta function is defined as

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n). \tag{E.6}$$

The following identities are satisfied by the Jacobi theta functions:

$$2\eta^3(\tau) = \theta_2(\tau)\theta_3(\tau)\theta_4(\tau), \tag{E.7}$$

$$\theta_2^4(\tau) - \theta_3^4(\tau) + \theta_4^4(\tau) = 0. \tag{E.8}$$

Under modular transformations, these change according to

$$\begin{aligned} \theta_2(-1/\tau) &= (-i\tau)^{1/2}\theta_4(\tau), & \theta_2(\tau+1) &= e^{i\pi/4}\theta_2(\tau), \\ \theta_3(-1/\tau) &= (-i\tau)^{1/2}\theta_3(\tau), & \theta_3(\tau+1) &= \theta_4(\tau), \\ \theta_4(-1/\tau) &= (-i\tau)^{1/2}\theta_2(\tau), & \theta_4(\tau+1) &= \theta_3(\tau), \\ \eta(-1/\tau) &= (-i\tau)^{1/2}\eta(\tau), & \eta(\tau+1) &= e^{i\pi/12}\eta(\tau). \end{aligned} \tag{E.9}$$