Superconformal String Theory

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Superconformal Action

$$\mathcal{S} = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\Big\{\partial_\alpha X^\mu(\sigma)\partial^\alpha X_\mu(\sigma) - i\bar{\psi}^\mu(\sigma)\rho^\alpha\partial_\alpha\psi_\mu(\sigma)\Big\}$$

$\psi^\mu_A \hookrightarrow$ Two-component worldsheet spinor, D-plet of Majorana fermion transforming in the vector representation of the Lorentz group $SO(D-1,1)$.

$\rho^\alpha_{AB} \hookrightarrow$ Two-dimensional Dirac matrix, satisfying the Clifford Algebra $\{\rho^\alpha,\rho^\beta\} = -2\eta^{\alpha\beta}$. The matrices ρ^α are chosen to be purely imaginary.

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

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Grassmann numbers

Grassmann numbers form a non-commutative ring with $\mathbb{Z}_2$ grading,

$$\text{Even} \rightarrow |\chi| = 0 \quad , \quad \text{Odd} \rightarrow |\chi| = 1$$

The product of two Grassmann numbers is commutative unless both factors are odd in which case it is anti-commutative

$$\chi\psi = (-1)^{|\chi||\psi|}\psi\chi$$

The coordinates of the bosonic string, $X^\mu(\sigma, \tau)$, are represented classically as commuting variables (even G).

The spinors, $\psi^\mu(\sigma, \tau)$, are represented classically as anticommuting variables (odd G).

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Supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu$$

$$\delta\psi^\mu = -i\rho^\alpha\partial_\alpha X^\mu\epsilon$$

ϵ is a constant, infinitesimal Majorana spinor (Odd G).

$$X^\mu \longleftrightarrow \psi^\mu$$

Supersymmetry transformations relates bosonic and fermionic coordinates!!

Commuting two supersymmetry transformations we get a worldsheet translation:

$$[\delta_1, \delta_2]X^\mu = a^\alpha\partial_\alpha X^\mu$$

$$[\delta_1, \delta_2]\psi^\mu = a^\alpha\partial_\alpha\psi^\mu$$

Note: the second equation holds only if ψ^μ is on-shell.

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The invariance of $\mathcal{S}$ under supersymmetry transformations implies, through the *Noether Theorem*, the existence of a conserved fermionic current.

Supercurrent

$$J_\alpha = \frac{1}{2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu$$

The invariance of the theory under translation on the worldsheet gives rise to another conserved current, the Stress-Energy Tensor.

Stress-Energy Tensor

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \frac{i}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - (trace)$$

Properties

Conservation	$\partial_\alpha T^{\alpha\beta} = 0$	Traceless	$T^\alpha_\alpha = 0$
	$\partial_\alpha J^\alpha = 0$		$\rho^\alpha J_\alpha = 0$

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Superspace

Supersymmetry can be made manifest through the introduction of a two dimensional superspace. In superspace, the worldsheet coordinates, σ^α, are supplemented by two anticommuting Grassmann coordinates θ^A.

Definition of Superfield

$$Y^\mu(\sigma,\theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma)$$

where B^μ is an auxiliary field. The generator of supersymmetry corresponds to the generator of translation in superspace

$$Q_A = \frac{\partial}{\partial\bar{\theta}^A} + i(\rho^\alpha\theta)_A\partial_\alpha$$

The supercharge generates the infinitesimal transformation of the superfield

$$\delta Y^\mu = [\bar{\epsilon} Q, Y^\mu] = \bar{\epsilon} Q Y^\mu$$

$$[\delta_1, \delta_2] Y^\mu = -a^\alpha \partial_\alpha Y^\mu$$

Supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu$$

$$\delta\psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon + B^\mu \epsilon$$

$$\delta B^\mu = -i\bar{\epsilon}\rho^\alpha \partial_\alpha \psi^\mu$$

The closure of the supersymmetry algebra is achieved thanks to the auxiliary field B^μ, whose vanishing accounts for the on-shell condition of the fermionic field.

Equations of motion

In light-cone coordinates ($\sigma^{\pm} = \tau \pm \sigma$) the fermionic part of the action results

$$\mathcal{S}_f = \frac{i}{\pi} \int \mathrm{d}^2\sigma \{\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+\}$$

where $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$ is a two component spinor.

The equations of motion as functions of the right and left moving components are

$$\partial_+ \psi_-^{\mu} = \partial_+ \left(\partial_- X^{\mu}\right) = 0$$
$$\partial_- \psi_+^{\mu} = \partial_- \left(\partial_+ X^{\mu}\right) = 0$$

The conserved currents in light-cone coordinates results

$$J_+ = \psi_+^\mu \partial_+ X_\mu$$
$$J_- = \psi_-^\mu \partial_- X_\mu$$

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{\mu +}$$
$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{\mu -}$$

Super-Virasoro algebra

$$\{J_-(\sigma), J_-(\sigma')\} = \pi\delta(\sigma - \sigma')T_{--}(\sigma)$$
$$\{J_+(\sigma), J_+(\sigma')\} = \pi\delta(\sigma - \sigma')T_{++}(\sigma)$$
$$\{J_+(\sigma), J_-(\sigma')\} = 0$$

Super-Virasoro constraints

$$T_{++} = T_{--} = J_+ = J_- = 0$$

$\hookrightarrow$ *local supersymmetry & gauge-invariant Lagrangian*

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Boundary conditions for Closed strings

Periodicity (R)	Antiperiodicity (NS)
$\psi_A^\mu(\sigma,\tau) = \psi_A^\mu(\sigma+\pi,\tau)$	$\psi_A^\mu(\sigma,\tau) = -\psi_A^\mu(\sigma+\pi,\tau)$

The antiperiodicity condition is due to the fact that ψ_A^μ, being a spinor on the worldsheet, can be itself or minus itself after a complete rotation around the string.

The general solutions of the Dirac equation ($\partial_+ \psi_-^\mu = 0$) in Fourier modes for the right-moving component are

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \quad (R)$$

or

$$\psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir(\tau-\sigma)} \quad (NS)$$

and for the left-moving component ($\partial_- \psi_+^\mu = 0$)

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \quad (R)$$

or

$$\psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)} \quad (NS)$$

Corresponding to the different pairings of (ψ_-^μ, ψ_+^μ) we obtain four closed-string sectors: (NS-NS), (NS-R), (R-NS), (R-R).

Boundary conditions for Open strings

The vanishing of the surface term derived from the variation of $\mathcal{S}$ in light-cone coordinates

$$\delta\mathcal{S} = \int d^2\sigma\delta\{\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+\}$$

requires

$$\psi_+(\sigma,\tau) = \pm\psi_-(\sigma,\tau) \qquad \sigma = 0,\pi$$

At one end of the string the relative sign can be chosen to be $\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau)$, whereas at the other end the sign acquires significance and defines two types of sectors:

Ramond (R)	Neveu-Schwarz (NS)
$\psi_+^\mu(\pi,\tau) = \psi_-^\mu(\pi,\tau)$	$\psi_+^\mu(\pi,\tau) = -\psi_-^\mu(\pi,\tau)$

The general solutions of the Dirac equation in Fourier modes result, with Ramond boundary condition,

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)}$$

$$\psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)},$$

with Neveu-Schwarz boundary condition,

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Superconformal modes

The Fourier modes of the conserved currents $T_{\alpha\beta}$ and J_α correspond to the super-Virasoro modes, for open strings

$$L_m = \frac{1}{\pi}\int_0^\pi d\sigma\{e^{im\sigma}T_{++} + e^{-im\sigma}T_{--}\}$$

$$F_m = \frac{\sqrt{2}}{\pi}\int_0^\pi d\sigma\{e^{im\sigma}J_{+} + e^{-im\sigma}J_{-}\}$$

$$G_r = \frac{\sqrt{2}}{\pi}\int_0^\pi d\sigma\{e^{ir\sigma}J_{+} + e^{-ir\sigma}J_{-}\}$$

For closed string there are two sets of super-Viraroso generators, one given by the mode expansions of T_{++} and J_+ whereas the other given by the mode expansions of T_{--} and J_-.

Two different quantization procedures

Covariant quantization : First the fields are promoted to operators and then, imposing the constraint equations on the states, the negative norm states are eliminated.

Light-Cone quantization: First find the space of physical states, fixing the light-cone gauge and solving the constraints, and after quantize the system.

The two methods should agree.

Covariant quantization

In order to quantize bosonic and fermionic coordinates in a two dimensional free field theory, X^μ and ψ^μ are promoted to operator valued fields obeying the following canonical commutation relations.

$$[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau,)] = -i\pi\eta^{\mu\nu}\delta(\sigma-\sigma')$$
$$\{\psi^\mu_A(\sigma,\tau), \psi^\nu_B(\sigma',\tau)\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma')$$

These equations imply the following relations:

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}$$
$$\{d^\mu_m, d^\nu_n\} = \delta_{m+n}\eta^{\mu\nu}$$
$$\{b^\mu_r, b^\nu_s\} = \delta_{r+s}\eta^{\mu\nu}$$

where $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$.

The oscillating modes become either annihilation operators, when the index is positive

$$\alpha_m^\mu |0\rangle = b_r^\mu |0\rangle = 0 \quad m, r > 0$$

$$\alpha_m^\mu |0\rangle = d_m^\mu |0\rangle = 0 \quad m > 0$$

or creation operators, when the index is negative. For $m, r < 0 \quad \alpha_m^\mu$, d_m^μ and b_r^μ increase the eigenvalue of M^2 by $2m$ and $2r$ units, respectively.

Half integer modes $\rightarrow$ unique non degenerate ground state.

Integer modes $\rightarrow$ the ground state is not uniquely defined, $[d_0^\mu, M^2] = 0$
Furthermore d_0^μ form the Clifford algebra: $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$.

NS-sector $\longrightarrow$ the states are spacetime bosons
R-sector $\longrightarrow$ the states are spacetime fermions.

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Generators of the superconformal algebra

Fermionic sector - (R)

$$L_m = L_m^\alpha + L_m^d$$

$$L_m^d = \frac{1}{2}\sum_{n\in\mathbb{Z}}(n + \frac{1}{2}m) : d_{-n}d_{m+n} : \qquad F_m = \sum_{n\in\mathbb{Z}} \alpha_{-n}d_{m+n}$$

Bosonic sector - (NS)

$$L_m = L_m^\alpha + L_m^b$$

$$L_m^b = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2}(r + \frac{1}{2}m) : b_{-r}b_{m+r} : \qquad G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n}b_{r+n}$$

where in both the two sectors

$$L_m^\alpha = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{-n}\alpha_{m+n} :$$

Quantizing the system, the algebra of the Fourier modes acquires a central extension and becomes the super-Virasoro algebra.

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n}$$

(NS)

$$[L_m, G_r] = \left(\tfrac{1}{2}m - r\right)L_{m+r}$$

$$\{G_r, G_s\} = 2L_{r+s} + B(r)\delta_{r+s}$$

(R)

$$[L_m, F_n] = \left(\tfrac{1}{2}m - n\right)L_{m+n}$$

$$\{F_m, F_n\} = 2L_{m+n} + B(m)\delta_{m+n}$$

Anomalies

(NS)

$$A(m) = \tfrac{1}{8}D(m^3 - m)$$

$$B(r) = \tfrac{1}{2}D(r^2 - \tfrac{1}{4})$$

(R)

$$A(m) = \tfrac{1}{8}Dm^3$$

$$B(r) = \tfrac{1}{2}Dm^2$$

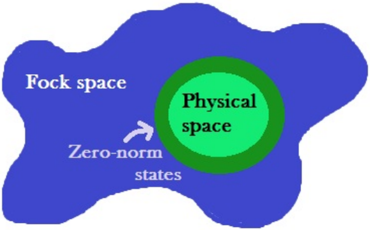
In the quantum theory the physical constraints become

$$L_n |\phi\rangle = 0 \quad n > 0 \qquad\qquad L_n |\psi\rangle = 0 \quad n > 0$$

$$(L_0 - a) |\phi\rangle = 0 \qquad\qquad (F_0) |\psi\rangle = 0$$

$$G_r |\phi\rangle = 0 \quad r > 0 \qquad\qquad F_n |\psi\rangle = 0 \quad n > 0$$

The Fock space built up by the oscillators α^μ_m, d^μ_m and b^μ_r is not positive definite. Only a subspace of the entire Fock space has this property.



Extra-physical states of zero-norm are found for $a = 1/2, 0$ in the bosonic and fermionic sector respectively, and critical spacetime dimension $D = 10$.

Light-cone gauge

The residual gauge freedom that arises from the symmetry of the system under conformal transformations can be used to make the following noncovariant choice,

$$X^{+}(\sigma,\tau) = x^{+} + p^{+}\tau$$

The same apply to the fermionic coordinates, but this time thanks to the freedom of applying local supersymmetry transformations that preserve the gauge choices

$$\psi^{+}(\sigma,\tau) = 0$$

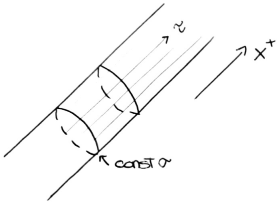


Figure : Every point on the string is at the same value of "time"

As check of consistency : $\delta X^{+} = \bar{\epsilon}\psi^{+} = 0$

The super-Virasoro constraints ($T_{++} = T_{--} = J_+ = J_- = 0$) in light-cone coordinates result

$$\psi_\pm \cdot \partial_\pm X = 0$$
$$(\partial_\pm X)^2 + \tfrac{i}{2}\psi_\pm \cdot \partial_\pm \psi_\pm = 0$$

From these expressions X^- and ψ^- are fixed from the following differential equations

$$\partial X^- = \frac{1}{p^+}(\partial X^i \cdot \partial X^i + \tfrac{i}{2}\psi^i \cdot \partial \psi^i)$$
$$\psi^- = \frac{2}{p^+}\psi^i \cdot \partial X^i$$

Leaving only the transverse oscillators X^i, ψ^i as free coordinates.

Having identified the physical degrees of freedom as the transverse oscillating modes, the next step consists in quantizing the system.
The canonical commutation relations for the transverse oscillators are

$$[\alpha^i_m, \alpha^j_n] = m\delta_{m+n}\delta^{ij}$$
$$\{d^i_m, d^j_n\} = \delta_{m+n}\delta^{ij}$$
$$\{b^i_r, b^j_s\} = \delta_{r+s}\delta^{ij}$$

and

$$\{x^-, p^+\} = -i$$

for the center of mass light-cone coordinates.
The fundamental canonical commutation relations arise from the negative longitudinal modes α^-_n, b^-_r

$$[p^+\alpha^-_m, p^+\alpha^-_n] = (m-n)p^+\alpha^-_{m+n} + \left[\frac{D-2}{8}(m^3 - m) + 2am\right]\delta_{m+n}$$

$$\{p^+b^-_r, p^+b^-_s\} = p^+\alpha^-_{r+s} + \left[\frac{D-2}{2}(r^2 - \tfrac{1}{4}) + 2a\right]\delta_{r+s}$$

The ligth-cone gauge manifestly breaks the Lorentz invariance of the theory. The transformations that do not preserve the gauge condition (J^{i-} and J^{+-}) could give rise to an anomaly term in the Lorentz algebra.
In fact, J^{i-} has commutation relations

$$[J^{i-}, J^{j-}] \neq 0$$

while for the Lorentz algebra

$$[J^{\mu\nu}, J^{\rho\lambda}] = -i\eta^{\nu\rho} J^{\mu\lambda} + i\eta^{\mu\rho} J^{\nu\lambda} + i\eta^{\nu\lambda} J^{\mu\rho} - i\eta^{\mu\lambda} J^{\nu\rho}$$

it should vanish.

The quantization of the system gives rise to an anomaly term in the Lorentz algebra.

$$[J^{i-}, J^{j-}] = (p^+)^2 \sum_{m=1}^{\infty} \Delta_m (\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m)$$

where

$$\Delta_m = m\left(1 - \frac{D-2}{8}\right) + \frac{1}{m}\left(\frac{D-2}{8} - 2a\right)$$

Thus, for general values of a and D the theory is not Lorentz invariant. Spacetime Lorentz symmetry is recovered constraining the two parameters

$$D = 10 \qquad \wedge \qquad a = \frac{1}{2}$$

Superstring Action

$$\mathcal{S} = \mathcal{S}' + \mathcal{S}''$$

$$\mathcal{S}' = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma e \Big\{ h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \Big\}$$

$$\mathcal{S}'' = -\frac{1}{\pi}\int \mathrm{d}^2\sigma e \Big\{ \bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \partial_\beta X_\mu + \tfrac{1}{4}\bar{\psi}^\mu \psi_\mu \bar{\chi}_\alpha \rho^\beta \rho^\alpha \chi_\beta \Big\}$$

Invariant under the local supersymmetry transformations

$$\delta X^\mu = \epsilon\psi^\mu, \qquad \delta\psi^\mu = -i\rho^\alpha \epsilon(\partial_\alpha X^\mu - \bar{\psi}^\mu \chi_\alpha)$$

$$\delta e^a_\alpha = -2i\bar{\epsilon}\rho^a \chi_\alpha, \qquad \delta\chi_\alpha = \nabla_\alpha \epsilon$$

From the locally supersymmetric action, the constraint equations for the conserved currents, $J_\alpha = 0$ and $T_{\alpha\beta} = 0$, are derived as equation of motions of the new field, χ_α, and of the world sheet metric, $h_{\alpha\beta}$.

Supercurrent

$$J_\alpha \equiv \frac{\pi}{2e}\frac{\delta \mathcal{S}}{\delta \chi^\alpha} = \tfrac{1}{2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu$$

Stress-Energy Tensor

$$T_{\alpha\beta} \equiv \frac{-2}{\pi\sqrt{h}}\frac{\delta \mathcal{S}}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu + \tfrac{i}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \tfrac{i}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - (trace)$$

The symmetries of the action can be used to impose the superconformal gauge. This amount to set the world-sheet metric as the flat Minkowski metric and to remove the gravitino field.

$$h^{\alpha\beta} = \eta^{\alpha\beta} \quad (e^a_\alpha = \delta^a_\alpha), \qquad\qquad \chi_\alpha = 0$$

In conclusion, from the locally symmetric action we can derive the gauge-fixed action

$$\mathcal{S} = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma \Big\{ \partial_\alpha X^\mu(\sigma)\partial^\alpha X_\mu(\sigma) - i\bar{\psi}^\mu(\sigma)\rho^\alpha\partial_\alpha\psi_\mu(\sigma) \Big\}$$

Conclusions

Results

- Presence of fermions
- Critical spacetime dimension $D = 10$
- Constant mass shift $a = \frac{1}{2}$.

Problem to be solved

- Presence of tachyons

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References

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